MAT2705-01/02 16F Quiz 9 Print Name (Last, First) ________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFYexpressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. $x_1' = 4x_1 + 10x_2,\ x_2' = -5x_1 - 6x_2,\ x_1(0) = 3,\ x_2(0) = 3$

a) Write down the Maple solution of this initial value problem, simpified to integer coefficients.

b) Rewrite this system of DEs **and** its initial conditions explicitly in matrix form for the vector variable $\vec{x} = \langle x_1, x_2 \rangle$ as a column matrix (using the actual matrices, not their symbols), identifying the coefficient matrix A.

c) Derive by hand its eigenvalues $\lambda_\pm = -k \pm I\omega$ and eigenvectors $\vec{b}_\pm$, $B = \langle \vec{b}_+ | \vec{b}_- \rangle$, and check that they agree with Maple.

d) Evaluate the real and imaginary parts of $\vec{z} = e^{\lambda_+ t}\vec{b}_+ = \vec{u} + I\vec{v}$.

e) Let $\vec{x} = c_1\vec{u} + c_2\vec{v}$. Solve the condition $\vec{x}(0) = \langle 3, 3 \rangle$ for (c_1, c_2), backsubstitute into $\vec{x}$ and simplify. Make sure that it agrees with part a).

f) Express the sinusoidal factor in each vector component of $\vec{x}$ in phase-shifted form $x_i = A_i e^{-kt}\cos(\omega t - \delta_i)$ to identify the exponential envelope functions. Back up your calculations with a common single diagram in the coefficient plane with the coefficient vectors of the two sinusoidal functions. Based on comparing the two phase shifts, which variable has its peaks shifted to the left of the other, x_1 or x_2?

g) Plot x_1 and x_2 versus t (use the original expressions, not the phase-shifted ones) for 5 characteristic times of the exponential factor starting at $t = 0$, including the envelopes of both decaying oscillations. Make a rough sketch of what you see, labeling the two curves. Is your claim in the previous part reflected in the plot?

h) Evaluate the polar form of the two components of $\vec{b}_+$ (namely $z_i = r_i e^{i\theta_i}$) and evaluate the difference $\theta_1 - \theta_2$.

i) **Optional.** What is the simplified ***exact*** phase shift angle $\delta_1 - \delta_2$ between the two solutions in radians? in degrees?

c) continued $B = \begin{bmatrix} -1-i & -1+i \\ 1 & 1 \end{bmatrix}$

▶ solution

1 a) $x_1 = e^{-t}(3\cos 5t + 9\sin 5t)$, $x_2 = e^{-t}(3\cos 5t - 6\sin 5t)$

b) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 10 \\ -5 & -6 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

c) $0 = |A - \lambda I| = \begin{vmatrix} 4-\lambda & 10 \\ -5 & -6-\lambda \end{vmatrix} = (\lambda-4)(\lambda+6) + 50 = \lambda^2 + 2\lambda - 24 + 50 = \lambda^2 + 2\lambda + 26$

$\boxed{\lambda_\pm} = \dfrac{-2 \pm \sqrt{4 - 4(26)}}{2} = -1 \pm \sqrt{-25} = \boxed{-1 \pm 5i} \rightarrow k = 1,\ \omega = 5$

$\lambda = -1 + 5i$:

$A - (-1+5i) = \begin{bmatrix} 4-(-1+5i) & 10 \\ -5 & -6-(-1+5i) \end{bmatrix} = \begin{bmatrix} 5-5i & 10 \\ -5 & -5-5i \end{bmatrix}$

$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (L F) $\quad x_2 = t$, $x_1 + (1+i)x_2 = 0$, $x_1 = -(1+i)t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(1+i)t \\ t \end{bmatrix} = t \begin{bmatrix} -(1+i) \\ 1 \end{bmatrix} \rightarrow \vec{b}_+$, $\quad \vec{b}_- = \begin{bmatrix} -(1-i) \\ 1 \end{bmatrix}$

d) $e^{\lambda_+ t}\vec{b}_+ = e^{-t}e^{5it}\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = e^{-t}(\cos 5t + i\sin 5t)\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

$= e^{-t}\begin{bmatrix} -\cos 5t + \sin 5t + i(-\cos 5t - \sin 5t) \\ \cos 5t + i\sin 5t \end{bmatrix}$

$= \underbrace{e^{-t}\begin{bmatrix} -\cos 5t + \sin 5t \\ \cos 5t \end{bmatrix}}_{\vec{X}_1} + i\,\underbrace{e^{-t}\begin{bmatrix} -\cos 5t - \sin 5t \\ \sin 5t \end{bmatrix}}_{\vec{X}_2}$

e) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} -\cos 5t + \sin 5t \\ \cos 5t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -\cos 5t - \sin 5t \\ \sin 5t \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \rightarrow c_2 = -6,\ c_1 = 3$

$e^{t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} -\cos 5t + \sin 5t \\ \cos 5t \end{bmatrix} - 6\begin{bmatrix} -\cos 5t - \sin 5t \\ \sin 5t \end{bmatrix}$

$= \begin{bmatrix} (-3+6)\cos 5t + (3+6)\sin 5t \\ 3\cos 5t - 6\sin 5t \end{bmatrix} = \begin{bmatrix} 3\cos 5t + 9\sin 5t \\ 3\cos 5t - 6\sin 5t \end{bmatrix}$ ✓

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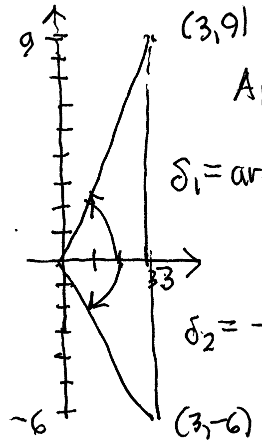
f)



$A_1 = 3\sqrt{1+3^2} = 3\sqrt{10}$

$\delta_1 = \arctan 3$

$\delta_2 = -\arctan 2$

$A_2 = 3\sqrt{1+2^2} = 3\sqrt{5}$

$$x_1 = 3\sqrt{10}\, e^{-t} \cos(5t - \arctan 3)$$
$$x_2 = 3\sqrt{5}\, e^{-t} \cos(5t + \arctan 2)$$

envelopes: x_1: $x = \pm 3\sqrt{10}\, e^{-t}$

x_2: $x = \pm 3\sqrt{5}\, e^{-t}$

[amplitude ratio $= \frac{\sqrt{10}}{\sqrt{5}} = \sqrt{2}$]

$\delta_1 > 0$ so x_1 is shifted right.

$\delta_2 < 0$ so x_2 is shifted left.

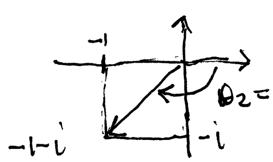
$\delta_1 - \delta_2$ is a positive obtuse angle

giving the phase that x_1 lags behind x_2 in time

(peaks to the right of x_2)

so x_2 peaks to the left of x_1

h) $\vec{b}_+ = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} e^{-\frac{3\pi}{4} i} \\ 1 e^{0i} \end{bmatrix}$ Note $\frac{r_1}{r_2} = \frac{\sqrt{2}}{1} = \sqrt{2}$

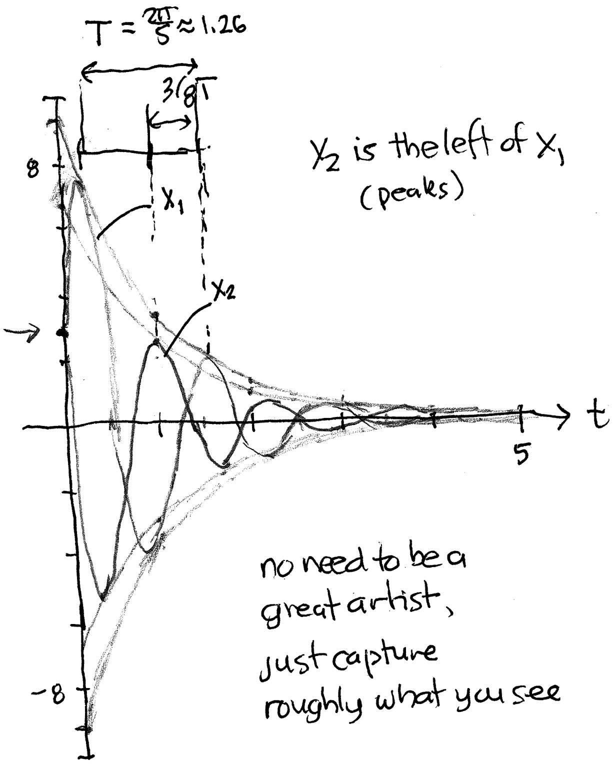


$\theta_2 = -\frac{3\pi}{4}$ since slope $= 1$

$A_2 = \sqrt{2}$

$$\theta_1 - \theta_2 = \theta_1 = -\frac{3\pi}{4}$$

g)



x_2 is the left of x_1 (peaks)

no need to be a great artist, just capture roughly what you see

i) $\delta_1 - \delta_2 = \arctan 3 - (-\arctan 2)$

$= \arctan 3 + \arctan 2 \approx 135.000000°$

$\left(= \pi + \arctan\left(\frac{3+2}{1-3\cdot 2}\right) = \arctan(-1) \right.$

$\left. = \pi + \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4} \right)$ trig magic!

so must be exactly $135° = 90° + 45°$

$\to \frac{3\pi}{4}$!

complex polar form easier than trig magic

Note $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{Re}\left(2c_1 e^{-t} e^{5it} \vec{b}_+\right) = a e^{-t} \mathrm{Re} \begin{bmatrix} A_1 e^{i(5t+\theta_1)} \\ A_2 e^{i(5t+\theta_2)} \end{bmatrix} = a e^{-t} \begin{bmatrix} A_1 \cos(5t - (\delta - \theta_1)) \\ A_2 \cos(5t - (\delta - \theta_2)) \end{bmatrix}$

where $2c_1 = a e^{-i\delta}$, $\vec{b}_+ = \begin{bmatrix} A_1 e^{i\theta_1} \\ A_2 e^{i\theta_2} \end{bmatrix}$

so $-\theta_i$ corresponds to relative phase shift