MAT2705-01/02 16F Final Exam Answers (1)

① a) $x_1 = \frac{35}{12}\cos t - \frac{11}{3}\cos 2t + \sin 2t + \frac{23}{4}\cos 3t - \frac{2}{3}\sin 3t$

$x_2 = \frac{245}{24}\cos t - \frac{22}{3}\cos 2t + 2\sin 2t - \frac{23}{8}\cos 3t + \frac{1}{3}\sin 3t$

$x_1(\pi) = -\frac{35}{12} - \frac{11}{3} - \frac{23}{4} = \frac{-35-\cancel{44}-69}{12} = -\frac{148}{12} = -\frac{37}{3}$ ✓

b) The cost terms are the response functions leaving the $\omega_1 = 2, \omega_2 = 3$ terms as the homogeneous soln so these are the "natural frequencies."

periods $T_i = 2\pi/\omega_i$: $(T_3, T_1, T_2) = (2\pi, \pi, \frac{2\pi}{3})$

$T = 2\pi$ is the common period containing (1, 2, 3) periods of these three periods.

c) $\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}''}_{\vec{x}''} = \underbrace{\begin{bmatrix} -8 & 2 \\ 2 & -5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 35\cos t \end{bmatrix}}_{\vec{F}}$

d) Eigenvectors (A):

$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \end{bmatrix}$ $\langle \vec{b_1} | \vec{b_2} \rangle = \begin{bmatrix} 1/2 & -2 \\ 1 & 1 \end{bmatrix}$ (or in opposite order)

e) $0 = |A - \lambda I| = \begin{vmatrix} -8-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = (\lambda+8)(\lambda+5) - 4$

$= \lambda^2 + 13\lambda + 36 \to \lambda = -4, -9$

$\lambda = -4$: $A + 4I = \begin{bmatrix} -8+4 & 2 \\ 2 & -5+4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{array}{cc} L & F \\ x_1 & x_2 \end{array}$

$x_2 = t$

$x_1 - x_2/2 = 0$

$x_1 = -\frac{1}{2}t$

$\langle x_1, x_2 \rangle = \langle \frac{1}{2}, t \rangle = \frac{t}{2} \underbrace{\langle 1, 2 \rangle}_{\vec{b_1}}$

$\lambda = -9$: $A + 9I = \begin{bmatrix} -8+9 & 2 \\ 2 & -5+9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{array}{cc} L & F \\ x_1 & x_2 \end{array}$

$x_2 = t$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2 = -2t$

$\langle x_1, x_2 \rangle = \langle -2t, t \rangle = t \underbrace{\langle -2, 1 \rangle}_{\vec{b_2}}$

$B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $B^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$A_B = B^{-1}AB = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix}$ Maple agrees!

f) $\vec{b_1} = \langle 1, 2 \rangle$, $m_1 = 2/1 = 2$

$\vec{b_2} = \langle -2, 1 \rangle$, $m_2 = 1/-2 = -1/2$

h) parallelogram vector addition

g) $\vec{y}(0) = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \to \vec{x}(0) = 1\vec{b_1} - 2\vec{b_2}$

$\vec{y}'(0) = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1'(0) \\ y_2'(0) \end{bmatrix} \to \vec{x}'(0) = 2\vec{b_1} + 1\vec{b_2}$

see sides of shaded parallelograms

$B^{-1}\vec{F} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 35\cos t \end{bmatrix} = \begin{bmatrix} 14\cos t \\ 7\cos t \end{bmatrix}$

i) $\vec{x}'' = A\vec{x} \underset{\vec{y} = B^{-1}x}{\xrightarrow{\vec{x} = B\vec{y}}} \vec{y}'' = A_B\vec{y}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 14\cos t \\ 7\cos t \end{bmatrix} = \begin{bmatrix} -4y_1 + 14\cos t \\ -9y_2 + 7\cos t \end{bmatrix}$

$y_1'' + 4y_1 = 14\cos t$

$y_{1h} = c_1\cos 2t + c_2\sin 2t$

$4\,[y_{1p} = c_5\cos t + c_6\sin t]$

$1\,[y_{1p}'' = -c_5\cos t - c_6\sin t]$

$y_{1p}'' + 4y_{1p} = (4-1)c_3\cos t + (4-1)c_4\sin t$

$= \underbrace{3c_3}_{=14}\cos t + \underbrace{3c_4}_{=0}\sin t = 14\cos t$

$c_3 = 14/3 \quad c_4 = 0$

$y_{1p} = 14/3 \cos t$

$y_2'' + 9y_2 = 7\cos t$

$y_{2h} = c_3\cos 3t + c_4\sin 3t$

$9\,[y_{2p} = c_7\cos t + c_8\sin t]$

$1\,[y_{2p}'' = -c_7\cos t - c_8\sin t]$

$y_{2p}'' + 9y_{2p} = (9-1)c_7\cos t + (9-1)c_8\sin t$

$= \underbrace{8c_7}_{=7}\cos t + \underbrace{8c_8}_{=0}\sin t = 7\cos t$

$c_7 = 7/8, \; c_8 = 0$

$y_{2p} = 7/8 \cos t$

$\boxed{\begin{aligned} y_1 &= c_1\cos 2t + c_2\sin 2t + 14/3\cos t \\ y_2 &= c_3\cos 3t + c_4\sin 3t + 7/8\cos t \end{aligned}}$ gen soln in new variables

j) see page 2 for differentiation step, then:

$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = B\begin{bmatrix} c_1 + 14/3 \\ c_3 + 7/8 \end{bmatrix} \to \begin{bmatrix} c_1 + 14/3 \\ c_3 + 7/8 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$c_1 = 1 - 14/3 = -11/3$, $c_3 = -2 - 7/8 = -23/8$

$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = B\begin{bmatrix} y_1'(0) \\ y_2'(0) \end{bmatrix} = B\begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} \to \begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$c_2 = 2/2 = 1$, $c_4 = 1/3$

$\boxed{\begin{aligned} y_1 &= -\tfrac{11}{3}\cos 2t + \sin 2t + \tfrac{14}{3}\cos t \\ y_2 &= \underbrace{-\tfrac{23}{8}\cos 3t + \tfrac{1}{3}\sin 3t}_{y_{ih}} + \underbrace{\tfrac{7}{8}\cos t}_{y_{ip}} \end{aligned}}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 - 2y_2 \\ 2y_1 + y_2 \end{bmatrix} =$

$= \begin{bmatrix} -\frac{11}{3}\cos 2t + \sin 2t + \frac{23}{4}\cos 3t - \frac{2}{3}\sin 3t + \underbrace{(\frac{14}{3} - \frac{7}{4})}_{=\frac{35}{12}}\cos t \\ -\frac{22}{3}\cos 2t + 2\sin 2t + -\frac{23}{8}\cos 3t + \frac{1}{3}\sin 3t + \underbrace{(\frac{28}{3} + \frac{7}{8})}_{=\frac{245}{24}}\cos t \end{bmatrix}$

Maple arithmetic

Maple agrees

MAT2705-01/02 16F FinalExam Answers (2)

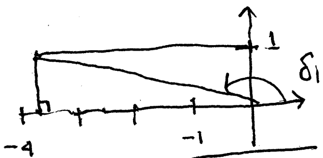
j) continued

$$\boxed{\begin{aligned} x_1 &= -\tfrac{11}{3}\cos 2t + \sin 2t + \tfrac{23}{4}\cos 3t - \tfrac{2}{3}\sin 3t + \tfrac{35}{12}\cos t \\ x_2 &= -\tfrac{22}{3}\cos 3t + 2\sin 2t - \tfrac{23}{8}\cos 3t + \tfrac{1}{3}\sin 3t + \tfrac{245}{24}\cos t \end{aligned}}$$

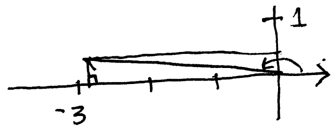
success!

k) $$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\left(-\tfrac{11}{3}\cos 2t + \sin 2t\right)}_{y_{1h}} \underset{\uparrow \text{ tandem}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \underbrace{\left(-\tfrac{23}{8}\cos 3t + \tfrac{1}{3}\sin 3t\right)}_{y_{2h}} \underset{\uparrow \text{ accordian}}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}} + \cos t \underbrace{\begin{bmatrix} 35/12 \\ 245/24 \end{bmatrix}}_{\vec{b}_3}$$

tandem (samesign)



$$\delta_1 = \pi - \arctan \frac{1}{11/3} = \pi - \arctan 3/11$$



$$\delta_2 = \pi - \arctan \frac{1/3}{23/8} = \pi - \arctan \frac{8}{69}$$

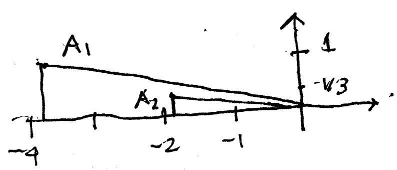
$$A_1 = \sqrt{\left(-\tfrac{11}{3}\right)^2 + 1^2} = \frac{\sqrt{130}}{3} \approx 3.8006$$

$$\boxed{A_i \cos(\omega_i t - \delta)}$$

$$A_2 = \sqrt{\left(-\tfrac{23}{8}\right)^2 + \left(\tfrac{1}{3}\right)^2} = \frac{5}{24}\sqrt{193} \approx 2.8943$$

$$\boxed{y_{1h} = \frac{\sqrt{130}}{3}\cos\left(2t - \pi + \arctan\tfrac{3}{11}\right)}$$

$$\boxed{y_{2h} = \frac{5}{24}\sqrt{193}\cos\left(3t - \pi + \arctan\tfrac{8}{69}\right)}$$



$\delta_2 > \delta_1$

$$A_1\vec{b}_1 \approx \begin{bmatrix} 3.80 \\ 7.60 \end{bmatrix} \qquad A_2\vec{b}_2 \approx \begin{bmatrix} -5.79 \\ 2.89 \end{bmatrix} \qquad \vec{b}_3 \approx \begin{bmatrix} 2.92 \\ 10.21 \end{bmatrix}$$

j) precede steps here by:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1\cos 2t + c_2 \sin 2t + 14/3 \cos t \\ c_3 \cos 3t + c_4 \sin 3t + 7/8 \cos t \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2c_1 \sin 2t + 2c_2 \cos 2t - 14/3 \sin t \\ -3c_3 \sin 3t + 3c_4 \cos 3t - 7/8 \sin t \end{bmatrix}$$

coefficient $\vec{b}_3$ and the homogeneous solution $\vec{x}_h$ (first two terms), as well as the final expressions for the two decoupled variables y_{1h} and y_{2h}. Which homogeneous term is associated with the tandem mode and which with the accordian mode? Is the response term a tandem or accordian mode?

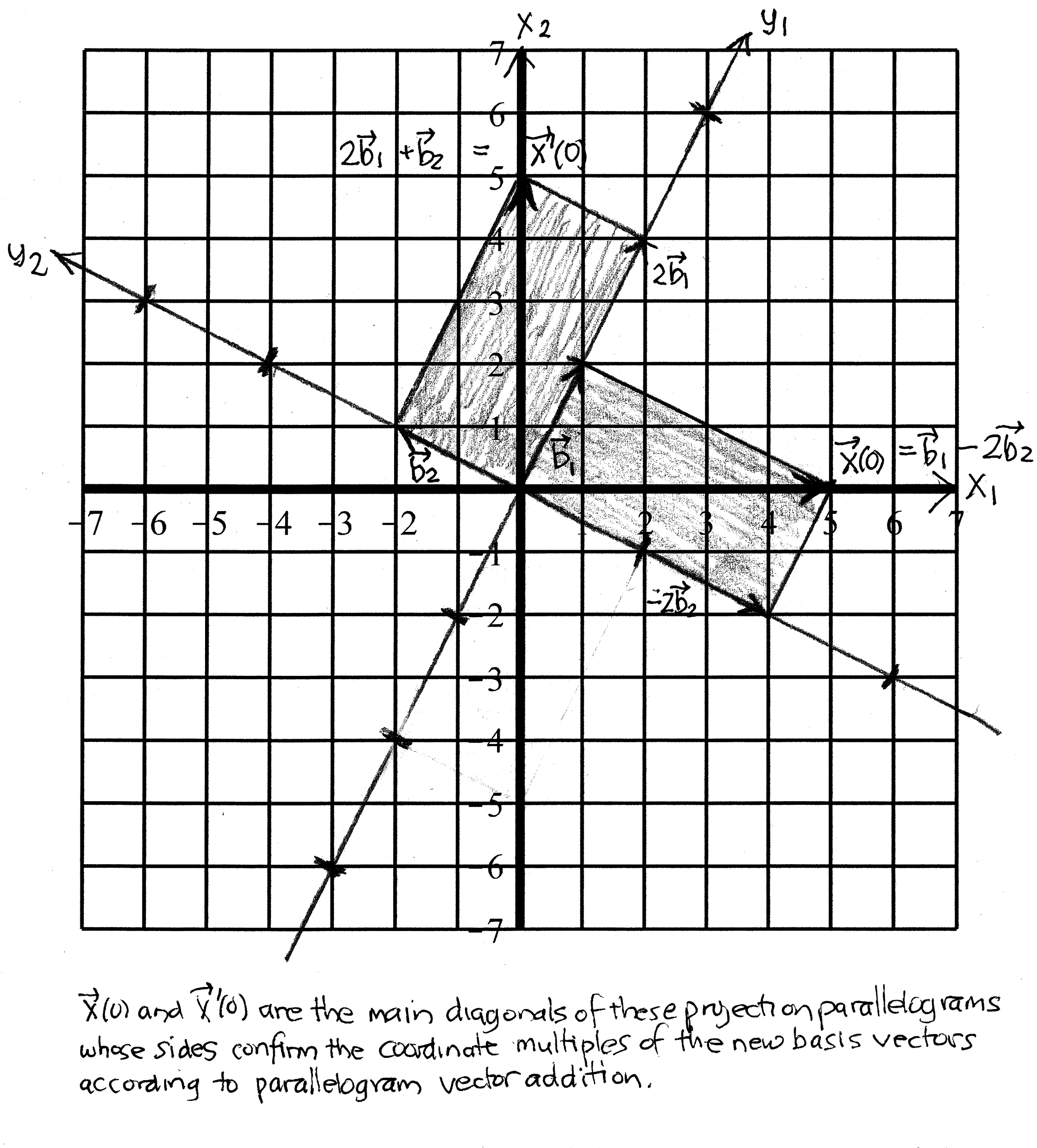
l) Write each of these sinusoidal functions y_{1h} and y_{2h} in phase-shifted cosine form stating explicitly (A_1, δ_1) and (A_2, δ_2) respectively, making a single completely labeled diagram in the sinusoidal coefficient plane that supports your work for each case. Evaluate the two vectors $A_1 \vec{b}_1$ and $A_2 \vec{b}_2$ as well as $\vec{b}_3$ numerically to 2 decimal places.

solution

pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have not accessed any of the class web pages or any other sites during the exam. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date:



$\vec{x}(0)$ and $\vec{x}'(0)$ are the main diagonals of these projection parallelograms whose sides confirm the coordinate multiples of the new basis vectors according to parallelogram vector addition.