MAT2705-02/05 15S Quiz 9 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). You may use technology to give the RREF form of matrices.

1.
$x_1'(t) = 15\,x_1(t) + 12\,x_2(t) - 12\,x_3(t)$, $x_2'(t) = -8\,x_1(t) - 7\,x_2(t) + 16\,x_3(t)$, $x_3'(t) = 4\,x_1(t) + 8\,x_2(t) + x_3(t)$, $x_1(0) = 6$, $x_2(0) = 12$, $x_3(0) = 6$

a) Rewrite this system of DEs **and** its initial conditions explicitly in matrix form for the vector variable $\boldsymbol{x} = \langle x_1, x_2, x_3\rangle$ as a column matrix (using the actual matrices, not their symbols), identifying the coefficient matrix A and suppressing function notation in the DE [x_1' not $x_1'(t)$].

b) For this A, using Maple write down the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (ordered by increasing value, they are integers!) and corresponding matrix of eigenvectors $B_{maple} = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle$ that it provides you, reordering them if necessary to order them as requested and rescaling their components to integer values if necessary; let B be this modified matrix.

c) Use technology to evaluate and write down the inverse matrix B^{-1} and use Maple to evaluate the matrix product $A_B = B^{-1} A\, B$. Write down this result. Does it evaluate correctly to the diagonalized matrix with the eigenvalues in the correct order?

d) Given that $\vec{x} = B\,\vec{y}$, if $\vec{x}\,(0) = \langle 6, 12, 6\rangle$, use the inverse matrix to find $\vec{y}\,(0)$, showing the matrix multiplication steps by hand.

e) Write down Maple's solution of this IVP for $\vec{x}\,(t)$ and compare it to the linear combination:
$y_1(0)\, e^{\lambda_1 t}\, \vec{b}_1 + y_2(0)\, e^{\lambda_2 t}\, \vec{b}_2 + y_3(0)\, e^{\lambda_3 t}\, \vec{b}_3$. Do they agree? [The correct solution has only integer parameters, so if you don't find this, perhaps you entered the coefficients incorrectly.]

▶ solution

① a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 15 & 12 & -12 \\ -8 & -7 & 16 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix}$

Note that: $\vec{y}\,' = A_B \vec{y}$
$y_1' = -9y_1$, $y_1 = c_1 e^{-9t}$
$y_2' = 9y_2$, $y_2 = c_2 e^{9t}$
$y_3' = 9y_3$, $y_3 = c_3 e^{9t}$

b) Maple: $\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -9 \\ 9 \\ 9 \end{bmatrix}$, $B_{maple} = \begin{bmatrix} 3/2 & 2 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \to B = \begin{bmatrix} 3 & 2 & -2 \\ -4 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

c) $B^{-1} = \frac{1}{9}\begin{bmatrix} -1 & -2 & 2 \\ 2 & 4 & 5 \\ -4 & 1 & 8 \end{bmatrix}$ $\quad A_B = B^{-1}AB = \begin{bmatrix} -9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ via Maple, correct.

d) $\vec{x}(0) = B\vec{y}(0)$, $\vec{y}(0) = B^{-1}x(0) = B^{-1}\langle 6,12,6\rangle \overset{\text{maple}}{=} \langle -2, 10, 4\rangle$

by hand: $\frac{1}{9}\begin{bmatrix} -1 & -2 & 2 \\ 2 & 4 & 5 \\ -4 & 1 & 8 \end{bmatrix}\begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix} = \frac{6}{9}\begin{bmatrix} -1(1) - 2(2) + 2(1) \\ 2(1) + 4(2) + 5(1) \\ -4(1) + 1(2) + 8(1) \end{bmatrix} = \frac{2}{3}\begin{bmatrix} -3 \\ 15 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 4 \end{bmatrix}$ ✓

e) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6e^{-9t} + 12e^{9t} \\ 8e^{-9t} + 4e^{9t} \\ -4e^{-9t} + 10e^{9t} \end{bmatrix}$ Maple

$\underbrace{\underbrace{y_1(0)}_{-2} e^{-9t}}_{y_1}\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} + \underbrace{\underbrace{y_2(0)}_{10} e^{9t}}_{y_2}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \underbrace{\underbrace{y_3(0)}_{4} e^{9t}}_{y_3}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6e^{-9t} + (20-8)e^{9t} \\ 8e^{-9t} + (0+4)e^{9t} \\ -2e^{-9t} + (10+0)e^{9t} \end{bmatrix} = \begin{bmatrix} -6e^{-9t} + 12e^{9t} \\ 8e^{-9t} + 4e^{9t} \\ -2e^{-9t} + 10e^{9t} \end{bmatrix}$ agree ✓