MAT2705-02/05 15S Quiz 6   Print Name (Last, First) ______________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **[You should use technology for row reductions and determinants. Report results you find in this way: reduced matrix, value of det.]**

1. a) Find the general solution of the following linear system, identifying your coefficient matrix A, the right hand side vector $\vec{b}$, the augmented matrix $\langle A|\,\vec{b}\rangle$, the RREF matrix and give the solution in scalar form: $x_1 = \ldots$, $x_2 = \ldots$ *etc.*.

$$3x_1 + 5x_2 - 7x_3 - x_4 = 8$$
$$x_1 + 2x_2 - 3x_3 - x_4 = 3$$
$$2x_1 - x_2 + 4x_3 + 8x_4 = 1$$

b) Write the solution in the column matrix vector form $\vec{x} = \vec{x}_{part} + \vec{x}_{hom}$, where the first term is a particular solution and the second term is the general solution of the corresponding homogeneous system with $\vec{b} = \vec{0}$.

c) Identify a basis of the solution space of the related homogeneous system. [Write your basis using curly brace set notation, using either angle bracket vector notation or column matrix notation.]

d) Consider the set of vectors $\{\vec{v}_1, \ldots, \vec{v}_4\}$ which form the columns of A. What are the independent relationships among these vectors that follow immediately from the reduced form of the matrix? (Rewrite each such relationship in the form: a linear combination of them equals the zero vector.)

e) Identify a basis of the span of the set of vectors $\{\vec{v}_1, \ldots, \vec{v}_4\}$ from the leading columns of A.

2. No two of the following sets of three vectors are proportional, so at most there might be one linear relationship among them. If appropriate, use the determinant to determine if they are linearly independent, justifying your conclusion. If linearly dependent, use the row reduced echelon form of their matrix of corresponding columns to write down single independent relationship among them and rewrite it in the form of an integer coefficient linear combination of these vectors equaling the zero vector. Double check that you have entered the vectors correctly.

a) $\{\langle 3, 0, 1, 2\rangle, \langle 1, -1, 0, 1\rangle, \langle 1, 2, 1, 0\rangle\}$

b) $\{\langle 1, -4, 0\rangle, \langle 4, 5, -1\rangle, \langle 0, -2, 5\rangle\}$,

c) $\{\langle 2, 0, -3\rangle, \langle 4, -5, -6\rangle, \langle -2, 1, 3\rangle\}$,

▶ solution

① a)
$$\underbrace{\begin{bmatrix} 3 & 5 & -7 & -1 \\ 1 & 2 & -3 & -1 \\ 2 & -1 & 4 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$$\langle A|\vec{b}\rangle = \begin{bmatrix} 3 & 5 & -7 & -1 & 8 \\ 1 & 2 & -3 & -1 & 3 \\ 2 & -1 & 4 & 8 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccccc} L & L & F & F & \ \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \ \end{array} \end{array}$$

$x_3 = t_1, x_4 = t_2$: $x_1 = -t_1 - 3t_2$, $x_2 = 2t_1 + 2t_2 - 1$

so $\boxed{x_1 = 1 - t_1 - 3t_2,\ x_2 = 1 + 2t_1 + 2t_2,\ x_3 = t_1,\ x_4 = t_2}$

b)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - t_1 - 3t_2 \\ 1 + 2t_1 + 2t_2 \\ t_1 \\ t_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_{part}} + \underbrace{t_1 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x}_{hom}}$$

c) $\vec{u}_1 = \langle -1, 2, 1, 0\rangle$, $\vec{u}_2 = \langle -3, 2, 0, 1\rangle$, $\boxed{\{\vec{u}_1, \vec{u}_2\} \text{ is a basis of the soln space}}$

(coefficient vectors of t_1, t_2)

d) $\vec{u}_1$ and $\vec{u}_2$ are coefficient vectors of lin. relationships among $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ if $A = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 | \vec{v}_4 \rangle$.

$\vec{u}_1$: $\boxed{-\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}}$ $\vec{u}_2$: $\boxed{-3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_4 = \vec{0}}$

alternate soln.

or 3rd col = (first col) − twice (second col) $\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2 \rightarrow -\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = 0$

4th col = 3 (first col) − twice (second col) $\vec{v}_4 = 3\vec{v}_1 - 2\vec{v}_2 \rightarrow -3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_4 = 0$

e) leading columns $\{\vec{v}_1, \vec{v}_2\}$ are a basis of span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

② a)
$$\begin{array}{c} \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \\ \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{v}_3 = 1\vec{v}_1 - 2\vec{v}_2$ so $\boxed{-\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = 0}$

b)
$$\begin{vmatrix} 1 & 4 & 0 \\ -4 & 5 & -2 \\ 0 & -1 & 5 \end{vmatrix} = 103 \neq 0 \text{ so cols are } \boxed{\text{lin. ind.}}$$

c)
$$\begin{array}{c} \begin{vmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{vmatrix} \\ \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \end{array} = 0 \quad \begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{v}_3 = -\frac{3}{5}\vec{v}_1 - \frac{1}{5}\vec{v}_2 \longrightarrow 5\vec{v}_3 = -3\vec{v}_1 - \vec{v}_2 \rightarrow \boxed{3\vec{v}_1 + \vec{v}_2 + 5\vec{v}_3 = \vec{0}}$

(see page 2)

(2) a) $A = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \underbrace{\begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ L \;\; L \;\; F \end{array}}_{\text{only coefficient matrix}}$

last column gives coeffs of leading cols to express it:

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

therefore:

$$\vec{v}_3 = 1\vec{v}_1 - 2\vec{v}_2$$

since old cols & new cols have same relationships!

putting all vectors on same side:

$$-\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}$$

to test $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for linear independence:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$$A\vec{x} = \vec{0}$$

$$\langle A | \vec{0} \rangle = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{cccc} L & L & F & \\ x_1 & x_2 & x_3 & \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

augmented matrix

$x_3 = t$: $\quad x_1 + x_3 = 0 \rightarrow x_1 = -t$
$\qquad\qquad x_2 - 2x_3 = 0 \rightarrow x_2 = 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$A\vec{x} = \vec{0}$ becomes with $t=1$:

$$-\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}$$

same result