MAT2705-02/05 15S TakeHomeTest 3 Answers (1)

① a) $\dfrac{9x''+6x'+10x=F}{9}$

$$x''+\underbrace{\tfrac{2}{3}}_{k_0}x'+\underbrace{\tfrac{10}{9}}_{\omega_0^2}x=\tfrac{1}{9}F$$

$$\boxed{\begin{array}{llll} k_0=2/3, & \tau_0=1/k_0=3/2, & \omega_0=\sqrt{\tfrac{10}{9}}=\tfrac{\sqrt{10}}{3} & T_0=\tfrac{2\pi}{\omega_0} \\ \approx 0.667 & =1.5 & \approx 1.054 & =\tfrac{2\pi\sqrt{3}}{\sqrt{10}} \\ Q=\omega_0\tau_0=\tfrac{3}{2}\tfrac{\sqrt{10}}{3}=\tfrac{\sqrt{10}}{2}\approx 1.581 & & & \approx 5.961 \end{array}}$$

b) $9x''+6x'+10x=0$

$x=e^{rt} \rightarrow (9r^2+6r+10)e^{rt}=0$

$9r^2+6r+10=0$

$r=\dfrac{-6\pm\sqrt{36-4\cdot 9\cdot 10}}{2\cdot 9}=-\dfrac{1}{3}\pm\dfrac{6\cdot 3 i}{18}$

$=-\frac{1}{3}\pm i = -k_1\pm i\omega_1$

$$\boxed{\begin{array}{l} \omega_1=1,\quad \tau_1=1/(1/3)=3 \\ T_1=2\pi/\omega_1=2\pi\approx 6.283 \end{array}}$$

$e^{rt}=e^{(-\frac{1}{3}\pm i)t}=e^{-t/3}e^{\pm it}$

$=e^{-t/3}(\cos t\pm i\sin t)$

gen soln: $\boxed{x=e^{-t/3}(c_1\cos t+c_2\sin t)}$

c) $x'=-\frac{1}{3}e^{-t/3}(c_1\cos t+c_2\sin t)$

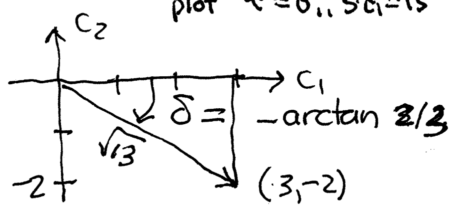
$+e^{-t/3}(-c_1\sin t+c_2\cos t)$

$x(0)=c_1 \qquad =3$

$x'(0)=-\frac{1}{3}c_1+c_2=-3\rightarrow c_2=-3+\frac{1}{3}(3)=-2$

$\boxed{x=e^{-t/3}(3\cos t-2\sin t)}$ IVP soln

plot $t=0..5\tau_1=15$



$\boxed{x=\underbrace{\sqrt{13}e^{-t/3}}\cos(t+\arctan 2/3)}$

$x=\pm\sqrt{13}\,e^{-t/3}$ envelope functions

d) $9x''+6x'+10x=17(e^{-t}-e^{-2t})$

$x_h=e^{-t/3}(c_1\cos t+c_2\sin t)$

$10\,[x_p=c_3e^{-t}+c_4e^{-2t}]$

$6\,[x_p'=-c_3e^{-t}-2c_4e^{-2t}]$

$9\,[x_p''=c_3e^{-t}+4c_4e^{-2t}]$

$9x_p''+6x_p'+10x_p=\underbrace{(10-6+9)}_{13}c_3e^{-t}+\underbrace{(10-12+36)}_{34}c_4e^{-2t} = 17(e^{-t}-e^{-2t})$

d) continued: $13c_3=17$, $34c_4=-17$

$c_3=\frac{17}{13}$, $\quad c_4=-\frac{1}{2}$

$x_p=\frac{17}{13}e^{-t}-\frac{1}{2}e^{-2t}$

$x=e^{-t/3}(c_1\cos t+c_2\sin t)+\frac{17}{13}e^{-t}-\frac{1}{2}e^{-2t}$

$x'=-\frac{1}{3}e^{-t/3}(c_1\cos t+c_2\sin t)-\frac{17}{13}e^{-t}+e^{-2t}$

$+e^{-t/3}(-c_1\sin t+c_2\cos t)$

$x(0)=c_1+\frac{17}{13}-\frac{1}{2}=0\rightarrow c_1=\frac{13-34}{26}=\frac{-21}{26}$

$x'(0)=-\frac{1}{3}c_1+c_2-\frac{17}{13}+1=0\rightarrow c_2=\frac{1}{3}\left(-\frac{21}{26}\right)+\frac{4}{13}=\frac{1}{26}$

$\boxed{x=\frac{1}{26}e^{-t/3}(\sin t-21\cos t)+\frac{17}{13}e^{-t}-\frac{1}{2}e^{-2t}}$

↳ plot, see max between 2 and 3, numerically solve $x'=0$ on this interval: $\boxed{\begin{array}{l} t_p\approx 2.4967 \\ x_p\approx 0.3952 \end{array}}$ peak values.

e) $9x''+6x'+10x=17\cdot 9\sin\omega t$

$10\,[x_p=c_3\cos\omega t+c_4\sin\omega t]$

$6\,[x_p'=-\omega c_3\sin\omega t+\omega c_4\cos\omega t]$

$9\,[x_p''=-\omega^2c_3\cos\omega t-\omega^2c_4\sin\omega t]$

$9x_p''+6x_p'+10x_p=[(10-9\omega^2)c_3+6\omega c_4]\cos\omega t$

$+[-6\omega c_3+(10-9\omega^2)c_4]\sin\omega t$

$=17\cdot 9\sin\omega t$

$(10-9\omega^2)c_3+6\omega c_4=0$

$-6\omega c_3+(10-9\omega^2)c_4=17\cdot 9$

$$\begin{bmatrix}10-9\omega^2 & 6\omega\\ -6\omega & 10-9\omega^2\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix}=\begin{bmatrix}0\\ 17\cdot 9\end{bmatrix}$$

$$\begin{bmatrix}c_3\\ c_4\end{bmatrix}=\frac{1}{(10-9\omega^2)^2+36\omega^2}\begin{bmatrix}10-9\omega^2 & -6\omega\\ 6\omega & 10-9\omega^2\end{bmatrix}\begin{bmatrix}0\\ 17\cdot 9\end{bmatrix}$$

$$=\frac{17\cdot 9}{(10-9\omega^2)^2+36\omega^2}\begin{bmatrix}-6\omega\\ 10-9\omega^2\end{bmatrix}$$

(*) steady state soln:

$$\boxed{x_p=\frac{17\cdot 9}{(10-9\omega^2)^2+36\omega^2}\left[-6\omega\cos\omega t+(10-9\omega^2)\sin\omega t\right]}$$

$\mathcal{D}=100+81\omega^4-144\omega^2$ expanded form

f) $A(\omega)=\dfrac{17\cdot 9}{(10-9\omega^2)^2+36\omega^2}\sqrt{36\omega^2+(10-9\omega^2)^2}=\dfrac{\overset{153}{17\cdot 9}}{\sqrt{(10-9\omega^2)^2+36\omega^2}}$

(*) Maple finds:

$x=153\omega\, e^{-t/3}(6\cos(t)+(9\omega^2-8)\sin t)/\mathcal{D}$ ↑ transient

$+x_p$ ↖ steady state

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1 f) continued. $A(0) = \frac{17.9}{10} = \frac{153}{10} = 15.3$

g) $0 = A'(\omega) = 17.9\left(-\frac{1}{2}\right)(\dots)^{-3/2}\left(\frac{4.81}{3^4}\omega^3 - \frac{2.144}{3^2 4^2}\omega\right)$

$\hookrightarrow \omega = 0,\ \omega^2 = \frac{2.144}{4.81} = \frac{8}{9}$

$\omega_p = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3} \approx .9428$

$A(\omega_p) = \frac{17.9}{\sqrt{\left(1-\frac{8}{9}\cdot 9\right)^2 + 36\cdot\frac{8}{9}}} = \frac{17.9}{\sqrt{4+32}} = \frac{17.9}{6} = \frac{51}{2} = 25.5$

$\frac{A(\omega_p)}{A(0)} = \frac{51/2}{153/10} = \frac{5}{3} \approx \underbrace{1.67 \gtrsim Q \approx 1.58}_{\text{peak ratio slightly bigger}}$

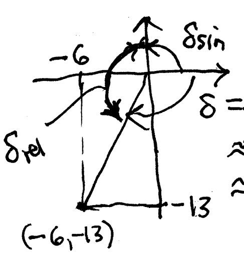
h) see plot, $\omega = 0..8$ or $\omega = 0..10$ shows asymptote at horizontal axis

i) $x = \underbrace{\frac{e^{-t/3}}{205}(2142\sin t + 459\cos t)}_{x_{\text{transient}}} - \underbrace{\frac{1}{410}(1989\sin 2t + 2459\cos 2t)}$

$x_{ss} = -\frac{153}{410}(13\sin 2t + 6\cos 2t)$

$A(2) = \frac{153}{410}\sqrt{13^2+6^2} = \frac{153}{2\sqrt{205}} \approx 5.343$

Note this agrees with our previous formula evaluated at $\omega = 2$.



$\delta = -\pi + \arctan 13/6$
≈ -2.003
$\approx -114.8°$

$\delta_{rel} = -\pi + \arctan 13/6 - \frac{\pi}{2} + 2\pi$
$= \frac{\pi}{2} + \arctan 13/6$
≈ 2.709 radians
$\approx 155.2°$
≈ 0.4311 cycles

j) $\tau_1 = 3,\ 5\tau_1 = 15$. see plot

2 a) $\frac{90}{3+6+6}\ \frac{\text{stuff}}{\text{unit vol}} = \boxed{6}\ \frac{\text{stuff}}{\text{unit vol}}$ = overall concentration

$6\langle 3,6,6\rangle = \boxed{\langle 18,36,36\rangle = \langle x_{1e}, x_{2e}, x_{3e}\rangle}$
V_1, V_2, V_3 — equilibrium distribution

b) $\langle k_1,k_2,k_3\rangle = \left\langle \frac{18}{3}, \frac{18}{6}, \frac{18}{6}\right\rangle = \langle 6,3,3\rangle$

$x_1' = -6x_1 + 3x_3$
$x_2' = 6x_1 - 3x_2$
$x_3' = 3x_2 - 3x_3$

$$\boxed{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}' = \underbrace{\begin{bmatrix}-6&0&3\\6&-3&0\\0&3&-3\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\ \begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix} = \begin{bmatrix}45\\0\\45\end{bmatrix}}$$

c) $0 = |A-\lambda I| = \begin{vmatrix}-6-\lambda&0&3\\6&-3-\lambda&0\\0&3&-3-\lambda\end{vmatrix} = -\lambda^3 - 12\lambda^2 - 45\lambda = -\lambda(\lambda^2+12\lambda+45)$

$\hookrightarrow \lambda = 0, -6\pm 3i$

2 c) continued.

$\lambda = 0$: $A - \lambda I = \begin{bmatrix}-6&0&3\\6&-3&0\\0&3&-3\end{bmatrix} \to \begin{bmatrix}1&0&-1/2\\0&-3&3\\0&3&-3\end{bmatrix} \to \begin{bmatrix}1&0&-1/2\\0&1&-1\\0&0&0\end{bmatrix}$

$\begin{bmatrix}1&0&-1/2\\0&1&-1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ L L F

$x_1 - \frac{1}{2}x_3 = 0 \to x_1 = \frac{1}{2}t$
$x_2 - x_3 = 0$, $x_2 = t$
$x_3 = t$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}t/2\\t\\t\end{bmatrix} = t\underbrace{\begin{bmatrix}+1/2\\1\\1\end{bmatrix}}_{\vec{b}_1}$

$\lambda = -6+3i$: $A - \lambda I = \begin{bmatrix}-6+6-3i&0&3\\6&-3+6-3i&0\\0&3&-3+6-3i\end{bmatrix}$

$= \begin{bmatrix}-3i&0&3\\6&3-3i&0\\0&3&3-3i\end{bmatrix} \to \begin{bmatrix}1&0&i\\0&3-3i&-6i\\0&1&1-i\end{bmatrix} \to \begin{bmatrix}1&0&i\\0&1&1-i\\0&0&0\end{bmatrix}$

$\begin{bmatrix}1&0&i\\0&1&1-i\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ L L F

$x_1 + ix_3 = 0$, $x_1 = -it$
$x_2 + (1-i)x_3 = 0$, $x_2 = -(1-i)t$
$x_3 = t$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-it\\-(1-i)t\\t\end{bmatrix} = t\underbrace{\begin{bmatrix}-i\\i-1\\1\end{bmatrix}}_{\vec{b}_2}$ $\quad \vec{b}_3 = \begin{bmatrix}i\\-i-1\\1\end{bmatrix}$ c.c.

$\vec{x} = c_1\vec{b}_1 + \underbrace{c_2 e^{(-6+3i)t}\vec{b}_2} + $ c.c.

$e^{-6t}(\cos 3t + i\sin 3t)\begin{bmatrix}-i\\i-1\\1\end{bmatrix} = z$

$= e^{-6t}\begin{bmatrix}\sin 3t - i\cos 3t\\ -\sin 3t - \cos 3t + i(\cos 3t - \sin 3t)\\ \cos 3t + i\sin 3t\end{bmatrix}$

$= \underbrace{e^{-6t}\begin{bmatrix}\sin 3t\\-\sin 3t - \cos 3t\\\cos 3t\end{bmatrix}}_{\vec{u}} + i\underbrace{e^{-6t}\begin{bmatrix}-\cos 3t\\\cos 3t - \sin 3t\\\sin 3t\end{bmatrix}}_{\vec{v}}$

$\boxed{\vec{x} = c_1\vec{b}_1 + c_2\vec{u} + c_3\vec{v}}$ gen. soln.

d) $\vec{x}(0) = c_1\vec{b}_1 + c_2\begin{bmatrix}0\\-1\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}45\\0\\45\end{bmatrix}$

$= \begin{bmatrix}+1/2&0&-1\\1&-1&1\\1&1&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$ Maple!

$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \underbrace{\begin{bmatrix}\downarrow\end{bmatrix}^{-1}}_{\frac{1}{5}\begin{bmatrix}2&2&2\\-2&-2&3\\-4&1&1\end{bmatrix}}\begin{bmatrix}45\\0\\45\end{bmatrix} = \begin{bmatrix}36\\9\\-27\end{bmatrix}$

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② d) continued

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 36\begin{bmatrix}1/2\\1\\1\end{bmatrix} + 9e^{-6t}\begin{bmatrix}S\\-S-C\\C\end{bmatrix} - 27e^{-6t}\begin{bmatrix}-C\\C-S\\S\end{bmatrix}$$

$$= \begin{bmatrix}18\\36\\36\end{bmatrix} + \begin{bmatrix}9S+27C\\18S-36C\\9C-27S\end{bmatrix}e^{-6t}$$

Maple agrees:

$$\boxed{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}18 + 9e^{-6t}\sin 3t + 27e^{-6t}\cos 3t\\36 + 18e^{-6t}\sin 3t - 36e^{-6t}\cos 3t\\36 + 9e^{-6t}\cos 3t - 27e^{-6t}\sin 3t\end{bmatrix}}$$

e) $\vec{x}_\infty = \langle 18, 36, 36\rangle \rightarrow C_\infty = \frac{x_{\infty i}}{V_i} = 6$!

f) $e^{-6t} \rightarrow \tau = 1/6$ agrees with a)

g) $t = 0..5/6$

$\langle c_1, c_2, c_3\rangle = \langle x_1/V_1, x_2/V_2, x_3/V_3\rangle$

$= \langle \frac{x_1}{3}, \frac{x_2}{6}, \frac{x_3}{6}\rangle$

③ a) $$\begin{bmatrix}x_1\\x_2\end{bmatrix}' = \underbrace{\begin{bmatrix}-22 & 12\\3 & -13\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\end{bmatrix} \qquad \begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = \begin{bmatrix}-3\\7\end{bmatrix}$$

$$0 = |A-\lambda I| = \begin{vmatrix}-22-\lambda & 12\\3 & -13-\lambda\end{vmatrix} = (\lambda+13)(\lambda+22) - 36$$

$$= \lambda^2 + 35\lambda + 250 \xrightarrow{\text{Maple}} \lambda = -10, -25$$

$$\lambda = -10: \; A + 10I = \begin{bmatrix}-22+10 & 12\\3 & -13+10\end{bmatrix} = \begin{bmatrix}-12 & 12\\3 & -3\end{bmatrix} \rightarrow \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$$

$$\underset{\text{L F}}{\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad x_1 - x_2 = 0 \quad x_1 = t \quad x_2 = t \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}t\\t\end{bmatrix} = t\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{b}_1}$$

$$\lambda = -25: \; A + 25I = \begin{bmatrix}25-22 & 12\\3 & 25-13\end{bmatrix} = \begin{bmatrix}3 & 12\\3 & 12\end{bmatrix} \rightarrow \begin{bmatrix}1 & 4\\0 & 0\end{bmatrix}$$

$$\underset{\text{L F}}{\begin{bmatrix}1 & 4\\0 & 0\end{bmatrix}}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad x_1 + 4x_2 = 0 \quad x_1 = -4t \quad x_2 = t \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-4t\\t\end{bmatrix} = t\underbrace{\begin{bmatrix}-4\\1\end{bmatrix}}_{\vec{b}_2}$$

$$\boxed{\begin{matrix}\lambda = -10, -25\\ B = \begin{bmatrix}1 & -4\\1 & 1\end{bmatrix}\end{matrix}} \quad \lambda_1 > \lambda_2 \quad \longrightarrow A_B = B^{-1}AB = \begin{bmatrix}-10 & 0\\0 & -25\end{bmatrix}$$

b) $$B^{-1} = \frac{1}{5}\begin{bmatrix}1 & 4\\-1 & 1\end{bmatrix} \qquad \vec{x} = B\vec{y}, \; \vec{y} = B^{-1}\vec{x}$$

$$\vec{x} = \begin{bmatrix}-3\\7\end{bmatrix} \rightarrow \vec{y} = B^{-1}\begin{bmatrix}-3\\7\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & 4\\-1 & 1\end{bmatrix}\begin{bmatrix}-3\\7\end{bmatrix} = \boxed{\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}y_1\\y_2\end{bmatrix}}$$

③ c) $\vec{x}' = A\vec{x} \leftarrow \vec{x} = B\vec{y}$

$$B^{-1}[(B\vec{y})' = AB\vec{y}]$$

$$\vec{y}' = A_B\vec{y}, \quad \begin{bmatrix}y_1\\y_2\end{bmatrix}' = \begin{bmatrix}-10 & 0\\0 & -25\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}-10y_1\\-25y_2\end{bmatrix}$$

$y_1' = -10y_1, \quad y_1 = c_1e^{-10t}$

$y_2' = -25y_2 \quad y_2 = c_2e^{-25t}$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = y_1\vec{b}_1 + y_2\vec{b}_2 = c_1e^{-10t}\begin{bmatrix}1\\1\end{bmatrix} + c_2e^{-25t}\begin{bmatrix}-4\\1\end{bmatrix}$$

$$\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = B\begin{bmatrix}y_1(0)\\y_2(0)\end{bmatrix} = B\begin{bmatrix}c_1\\c_2\end{bmatrix} \quad \text{so} \quad \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix} \text{ from b).}$$

$$\boxed{\begin{aligned}\begin{bmatrix}x_1\\x_2\end{bmatrix} &= 5e^{-10t}\begin{bmatrix}1\\1\end{bmatrix} + 2e^{-25t}\begin{bmatrix}-4\\1\end{bmatrix}\\ &= \begin{bmatrix}5e^{-10t} - 8e^{-25t}\\5e^{-10t} + 2e^{-25t}\end{bmatrix}\end{aligned}}$$

"vector form"

"scalar form"

e) $\tau_1 = \frac{1}{10} = .10 > \tau_2 = \frac{1}{25} = .04$

evaluate: $x_1(5\tau_1), x_2(5\tau_1)$ $\qquad 5\tau_1 = 1/2$

$$x_1(\tfrac{1}{2}) = 5e^{-10/2} - 8e^{-25/2} = 5e^{-5} - 8e^{-12.5} \approx 0.03366$$

$$x_2(\tfrac{1}{2}) = 5e^{-10/2} + 2e^{-25/2} = 5e^{-5} + 2e^{-12.5} \approx 0.03370$$

d) see plot.