MAT2705-02/05 15S Test 1 Print Name (Last, First) ________________________________|__

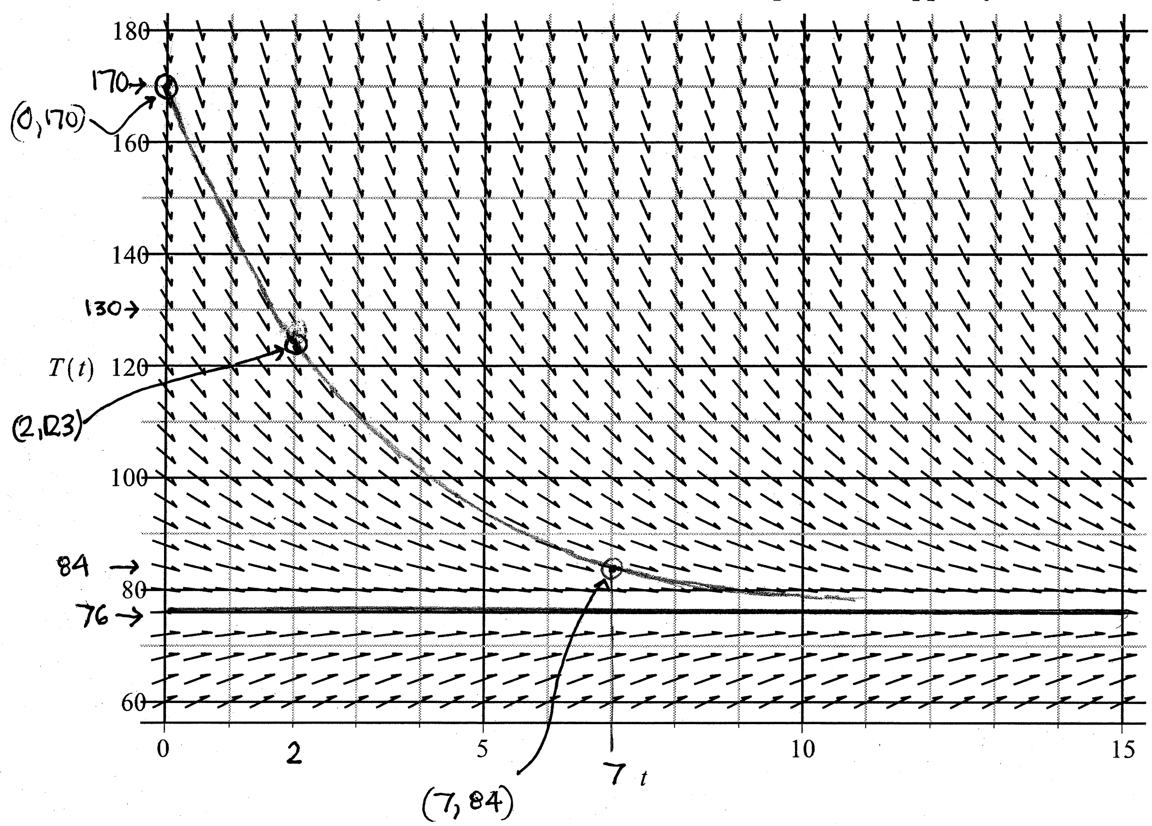
Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). [Recall you need $y'(t), y(t)$ instead of y', y in your differential equation for an unknown variable y for Maple to interpret the prime as a t derivative.]

1. $\frac{dT}{dt} = -k\,(T-A)$

a) A pot of liquid is put on the stove to boil. The temperature of the liquid reaches 170 degrees F and then the pot is taken off the burner and placed on a counter in the kitchen. The temperature of the air in the kitchen is 76 degrees F. After two minutes the temperature of the liquid in the pot is 123 degrees F. How long before the temperature of the liquid in the pot will be 84 degrees F? Assume this liquid obeys Newton's law of cooling. [Use the linear solution technique to solve this DE. Show every step of the process clearly. Answer this word problem with a complete English sentence which can be directly compared to a clock. Box it. During the process keep things exact and don't introduce any decimal points long enough for you reach an exact value for τ requested below. Use common sense about the number of significant digits you give the final answer.]

b) Using the slope field below, locate the initial data point and the secondary data point by circled dots. Then make a rough hand sketch of your solution, labeling on your sketch the initial and secondary data points given above. Is your hand drawn curve consistent with your answer to the question posed in part a) as well as the secondary information? Explain.

c) What is the exact value of the corresponding characteristic time τ (with units) for this exponential decay problem and its numerical value to 3 significant digits? How long does it take the temperature difference to drop to 5 percent of its initial value? How many characteristic times does this represent? Support your answer with a calculation.



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MAT 2705-02/05 15S Test 1 Answers

① a) $\frac{dT}{dt} = -k(T-A)$, $T(0) = T_0$

$A = 76$, $T_0 = 170$

$T(2) = 123$, $T(t) \stackrel{?}{=} 84$

solve:

$e^{kt}\left[\frac{dT}{dt} + kT = kA\right] \to \frac{d}{dt}\left(Te^{kt}\right) = kAe^{kt}$

$e^{\int k\,dt} = e^{kt}$

$Te^{kt} = \int kAe^{kt}\,dt = kA\frac{e^{kt}}{k} + C$

$T = e^{-kt}(Ae^{kt} + C) = A + Ce^{-kt}$ (gen soln)

init: $T(0) = \overset{76}{A} + C = 170 \to C = -76 + 170 = +94$

$T = 76 + 94e^{-kt}$ (IVP soln)

$T(2) = 76 + 94e^{-2k} = 123$

$e^{-2k} = \frac{123-76}{94} = \frac{47}{94}$, $e^{2k} = \frac{94}{47} = 2$

$k = \frac{1}{2}\ln 2 \approx 0.3466$

c) $\tau = 2/\ln 2 \approx 2.8854$

$\boxed{\approx 2.89}$ 3 sig. figs.

$T = 76 + 94e^{-kt} = 84$

$e^{-kt} = \frac{84-76}{94} = \frac{8}{94} = \frac{4}{47}$, $e^{kt} = \frac{47}{4}$

$t = \frac{1}{k}\ln\left(\frac{47}{4}\right) = 2\,\frac{\ln(47/4)}{\ln 2} \approx 7.109 \approx 7.11$

$.11(60) \approx 6.6 \approx 7$

so it will take about 7 minutes 7 seconds from the start time to cool down from 170°F to 84°F

c) $T - A = 94e^{-kt} = .05(94)$ temperature difference decays exponentially!

$e^{-kt} = .05$, $e^{kt} = 20$

$t = \frac{1}{k}\ln 20 = \tau \ln 20$

$\approx 8.644 \approx 2.9957\,\tau$

It takes about 8.64 minutes or 3.00 characteristic times for the temperature difference to drop to 5% of its original value of 94°F.

b) My hand drawn curve seems to pass through the point (7, 84) which is pretty close to $T(7.1) \approx 84$.

a) Maple: $\boxed{y = 1 - \sqrt{1 - x^2 + 4x}}$

② $\frac{dy}{dx} = \frac{2-x}{y-1}$, $y(0) = 0$ (separable)

b) $\int (y-1)\,dy = \int (2-x)\,dx$ u-sub more efficient

$\frac{(y-1)^2}{2} = -\frac{(x-2)^2}{2} + C_1$

see bottom for alternative soln!

$(y-1)^2 = -(x-2)^2 + 2C_1$, $2C_1 = C$

note: $(x-2)^2 + (y-1)^2 = C$ circle of radius $\sqrt{C}$ centered at (2,1)

solve for y: (avoids quadratic formula this way)

$y - 1 = \pm\sqrt{C - (x-2)^2}$

$\boxed{y = 1 \pm \sqrt{C - (x-2)^2}}$ gen soln two function graphs

c) init: $y = 0$ requires minus sign formula

set $x = 0, y = 0$:

$0 = 1 - \sqrt{C-4}$, $C - 4 = 1$, $C = 5$ (radius $\sqrt{5}$)

$\boxed{y = 1 - \sqrt{5 - (x-2)^2}}$

$= 1 - \sqrt{1 - x^2 + 4x}$ (expand square) (see below)

d) goes complex when

$5 - (x-2)^2 = 0$ (or $x^2 - 4x - 1 = 0$ ↳ quadratic formula)

$(x-2)^2 = 5$

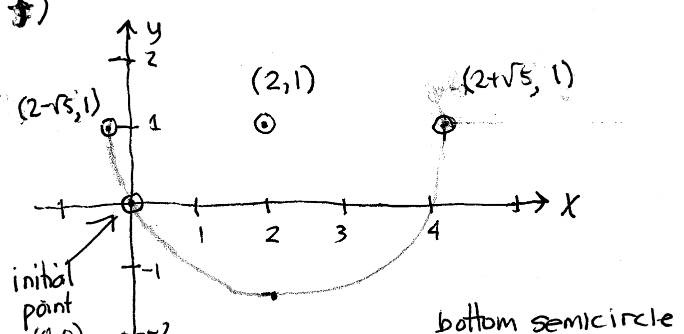
$x - 2 = \pm\sqrt{5}$, $x = 2 \pm \sqrt{5}$ exact!

so defined on interval:

$\boxed{\text{domain: } 2 - \sqrt{5} \le x \le 2 + \sqrt{5}}$ ends of diameter

≈ -0.236 ≈ 4.236

f)



$y = 1 - \sqrt{5 - (x-2)^2} = 1 - \sqrt{5 - (x^2 - 4x + 4)} = 1 - \sqrt{1 - x^2 + 4x}$

②a) alternative solution:

$\int (y-1)\,dy = \int (2-x)\,dx$ use quadratic formula!

$2\left[\frac{y^2}{2} - y = 2x - \frac{x^2}{2} + C_3\right]$ $y^2 - 2y + (x^2 - 4x - 2C_3) = 0$

$y = \frac{2 \pm \sqrt{4 - 4(x^2 - 4x - 2C_3)}}{2} = 1 \pm \sqrt{1 - (x^2 - 4x - 2C_3)}$

$0 = y(0) = 1 - \sqrt{1 + 2C_3} \to C_3 = 0$ (plus solution $y \ge 1$ so cannot work)

$y = 1 - \sqrt{1 - x^2 + 4x}$

(2e) $y = 1 - \sqrt{1 - x^2 + 4x} \to \frac{dy}{dx} = 0 - \frac{1}{2}\frac{(-2x+4)}{\sqrt{1-x^2+4x}} = \frac{x-2}{\sqrt{1-x^2+4x}}$ simplify

$\frac{dy}{dx} = \frac{2-x}{y-1}$ backsub $\to \frac{x-2}{\sqrt{1-x^2+4x}} = \frac{2-x}{1 - \sqrt{1-x^2+4x} - 1} = \frac{x-2}{\sqrt{1-x^2+4x}}$ ✓