MAT1505-03/04 15F TakeHomeTest 3 Answers (1)

① a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1-\frac{1}{n}}}{2^n} \underset{\text{behaves like}}{\overset{n\to\infty}{\sim}} \sum_{n=1}^{\infty} \frac{e^{1-0}}{2^n} = \sum_{n=1}^{\infty} e\left(\frac{1}{2}\right)^n$ geom series ratio $\frac{1}{2}<1$ converges

converges by limit comparison test

b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{n^2 2^{2n}} = \sum_{n=1}^{\infty} \frac{3}{n^2}\underbrace{\left(\frac{3}{4}\right)^n}_{r=3/4} \to \frac{a_{n+1}}{a_n} = \frac{3^{n+1+1}}{(n+1)^2 2^{2(n+1)}} \cdot \frac{n^2 2^{2n}}{3^{n+1} 2^{2n}} = \frac{3^{n+2-(n+1)}}{2^{2(n+1-n)}}\left(\frac{n}{n+1}\right)^2$

$\displaystyle = \frac{3}{2^2}\left(\frac{n}{n+1}\right)^2 \xrightarrow{n\to\infty} \frac{3}{4} < 1$ converges by ratio test.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{n}\sin\frac{1}{n} \underset{\text{behaves like}}{\overset{n\to\infty}{\sim}} \sum_{n=1}^{\infty} \frac{\pi}{n}\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{\pi}{n^2}$ proportional to $p=2$ series, converges since $p=2>1$ so

converges by limit comparison test

d) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$: $\displaystyle\frac{a_{n+1}}{a_n} = \frac{e^{(n+1)^2}}{(n+1)!}\,\frac{n!}{e^{n^2}} = \frac{n!}{(n+1)\,n!}\, e^{n^2+2n+1-n^2} = \frac{e^{2n+1}}{n+1}$

$\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{d}{dn}(e^{2n+1})}{\frac{d}{dn}(n+1)} = \lim_{n\to\infty} \frac{2e^{2n+1}}{1} = \infty$ diverges by ratio test

② $\sin 95^\circ = \sin\left(\frac{95\pi}{180}\right) = \sin\left(\frac{\pi}{2} + \frac{5\pi}{180}\right)$ suggests expanding about $x = \frac{\pi}{2}$

$f(x) = \sin x \quad f(\frac{\pi}{2}) = 1$ — even terms alternate in sign

$f'(x) = \cos x \quad f'(\frac{\pi}{2}) = 0$ — odd terms zero

$f''(x) = -\sin x \quad f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x \quad f'''(\frac{\pi}{2}) = 0$ etc

$\displaystyle\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}$

$\displaystyle = 1 - \frac{(x-\frac{\pi}{2})^2}{2} + \frac{(x-\frac{\pi}{2})^4}{4!} - \dots \quad (= \cos(x-\tfrac{\pi}{2})\,!)$ trig identity.

alternating series, next term is bound on error

$x - \frac{\pi}{2} = \frac{5\pi}{180} \equiv \alpha = \frac{\pi}{36}$

$\displaystyle\sin 95^\circ = 1 - \frac{1}{2}\alpha^2 + \frac{1}{24}\alpha^4 - \dots = \underbrace{1 - 0.003807718}_{.99619228} + \underbrace{0.000002416}_{\substack{\text{true value not greater by more}\\ \text{than } 2.4\times10^{-6}}} - 6.134\cdot10^{-6} + \dots$

sum of first 3 terms: 0.996194

If not clever to expand around $x=\pi/2$, the usual sine series requires the first 5 terms to get this accuracy.

$\approx \boxed{.996192}$

$\approx .99619$ ← error < 1 in this digit 9

← error certainly < 1 in this digit. ⇔ .0001

so $\boxed{0.99619}$ probably the best answer, keeping 1 more digit we know can't change

4 decimal place accuracy .9962

③ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3(x-2)^n}{2^{2n}(n+1)}$: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \frac{3|x-2|^{n+1}}{2^{2(n+1)}(n+1+1)}\,\frac{2^{2n}(n+1)}{3|x-2|^n} = 2^{2n-2(n+1)}\left(\frac{n+1}{n+2}\right)|x-3|^{n+1-n}$

$\displaystyle = \frac{1}{4}\left(\frac{n+1}{n+2}\right)|x-2| \xrightarrow[n\to\infty]{} \frac{1}{4}|x-3| < 1$ for absolute convergence by ratio test

$|x-2| < \underset{\substack{\parallel\\ R}}{4}$, $2-4 < x < 2+4$: $-2 < x < 6$ converges absolutely

R radius of convergence

$x-2=-4$, $x=-2$:

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3(-4)^n}{2^{2n}(n+1)} = \sum_{n=0}^{\infty} \frac{3(-1)^n(-1)^n 4^n}{4^n(n+1)} = \sum_{n=0}^{\infty} \frac{3}{n+1} \underset{n\to\infty}{\overset{\text{behaves like}}{\sim}} \sum_{n=1}^{\infty} \frac{3}{n}$ divergent $p=1$-series (harmonic) (p not >1)

diverges by limit comparison test

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③ continued) $x=6$, $x-2=4$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{2^{2n}(n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}$

alternating series, term decreasing to zero (in abs value) so converges by alternating series test.

so interval of convergence is $\boxed{-2 < x \le 4 \text{ or } (-2,4].}$

④ $\displaystyle f(x) = xe^x = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n x}{n!} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$

$$\int xe^x\,dx = \sum_{n=0}^{\infty}\frac{1}{n!}\int x^{n+1}dx = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{x^{n+2}}{n+2} + C$$

$$\int_0^1 xe^x\,dx = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 x^{n+1}dx = \sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{x^{n+2}}{n+2}\right|_0^1 = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{n+2}-0\right) = \sum_{n=0}^{\infty}\frac{1}{(n+2)\,n!}$$

$\|$

1 by int by parts done by Maple.

$$\therefore \sum_{n=0}^{\infty}\frac{1}{(n+2)\,n!} = 1$$

⑤ a) $\displaystyle\int\frac{1}{\sqrt{1-x^2}}\,dx = \underbrace{\arcsin(x)}_{\text{Maple}} + \underbrace{C}_{\text{us}}$ ✓

b) $\displaystyle(1-x^2)^{-1/2} = 1 - \tfrac{1}{2}(-x^2) - \frac{\frac{1}{2}\left(\frac{-3}{2}\right)}{1\cdot 2}(-x^2)^2 + \cdots = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \cdots$

$$\int(1-x^2)^{-1/2}dx = \int 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \cdots dx = \boxed{\underbrace{x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots}_{=\arcsin(x)}} + C$$

since $\arcsin(0)=0$

c) $f(x) = \arcsin(x)$ $\quad f(0)=0$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ $\quad f'(0)=1$

$f''(x) = \dfrac{x}{(1-x^2)^{3/2}}$ $\quad f''(0)=0$

$f'''(x) = \dfrac{2x^2+1}{(1-x^2)^{5/2}}$ $\quad f'''(0)=1$

$f^{(4)}(x) = \dfrac{3x(2x^2+3)}{(1-x^2)^{7/2}}$ $\quad f^{(4)}(0)=0$

$f^{(5)}(x) = \dfrac{3(8x^4+24x^2+3)}{(1-x^2)^{9/2}}$ $\quad f^{(5)}(0)=9$

so $\arcsin x = x + \underbrace{\frac{1}{3!}}_{6}x^3 + \underbrace{\frac{9}{5!}}_{\frac{9}{5\cdot4\cdot3\cdot2}=\frac{3}{40}}x^5 + \cdots$

$= x + \dfrac{x^3}{6} + \dfrac{3}{40}x^5 + \cdots$ ✓

Yes, Maple agrees.

⑥ a) $\displaystyle f(x) = (1+x)^{-2} = 1 - 2x - \frac{2(-3)}{1\cdot 2}x^2 - \frac{2(-3)(-4)}{1\cdot2\cdot3}x^3 - \frac{2(-3)(-4)(-5)}{1\cdot2\cdot3\cdot4}x^4$

$\displaystyle = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty}(n+1)(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

agree!

b) $\displaystyle (1+x)^{-2} = -\frac{d}{dx}(1+x)^{-1} = -\frac{d}{dx}\sum_{n=0}^{\infty}(-x)^n = \sum_{n=1}^{\infty}(-1)^{n+1}n\,x^{n-1} = \sum_{m=0}^{\infty}(-1)^m(m+1)x^m$

$m = n-1$
$n = m+1$

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(6) c) $F_{tidal} = \frac{GMm}{(r-R)^2} - \frac{GMm}{(r+R)^2} = \frac{GMm}{r^2}\left[\frac{1}{(1-R/r)^2} - \frac{1}{(1+R/r)^2}\right] = \frac{GMm}{r^2}\left[\left(1-\frac{R}{r}\right)^{-2} - \left(1+\frac{R}{r}\right)^{-2}\right]$

For better organization note:

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 - \cdots$$
$$(1-x)^{-2} - (1+x)^{-2} = 2\left[2x + 4x^3 + 6x^5 + \cdots\right]$$

so

$$F_{tidal} = 2\frac{GMm}{r^2}\left(2\frac{R}{r} + 4\left(\frac{R}{r}\right)^3 + \cdots\right) = \underbrace{\frac{4GMmR}{r^3}}_{a}\left(1 + \underbrace{\frac{2R^2}{r^2}}_{b} + \cdots\right)$$

evaluating in MKS units with at most 4 significant digits:

$$\boxed{\begin{array}{l} a \approx 1.313 \times 10^{19} \\ b \approx 5.494 \times 10^{-4} \end{array}}$$

optional fun

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}, \quad \text{let } x = \frac{1}{5},\ y = \frac{1}{239}$$

$$\pi = 4\left(\sum_{n=0}^{\infty} \frac{4(-1)^n x^{2n+1}}{2n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{2n+1}\right)$$

$$= 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\underbrace{(4x^{2n+1} - y^{2n+1})}_{>0}$$

alternating series, so next term in series is estimate of error in Taylor polynomial approximation

cutoff at $\boxed{n=4}$: 3.141591771 in Maple.

comparing to Maple's 10 digit value for π, the actual error is: -8.83×10^{-7} (a bit low) separable DEs are one step away from antidifferentiation.

$\frac{dy}{dx}$ = product or quotient of separate functions of x and y

for example $\left[\frac{dy}{dx} = 3xy\right]\frac{dx}{y}$ multiply both sides by dx to move it to RHS, divide both sides by y to move it to LHS

$\frac{dy}{y} = 3x\,dx$ variables have been "separated" to opposite sides of eqn

$\int \frac{dy}{y} = \int 3x\,dx$

$\ln|y| = 3\left(\frac{x^2}{2}\right) + C_1$ "implicit soln", try to solve for $y=y(x)$

$|y| = e^{\ln|y|} = e^{\frac{3}{2}x^2+C_1} = e^{C_1}e^{\frac{3}{2}x^2}$

$y = \underbrace{\pm e^{C_1}}_{\equiv C_1'} e^{\frac{3}{2}x^2} = C_1' e^{\frac{3}{2}x^2}$ general solution (explicit)

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another example: $\left[\frac{dy}{dx} = \frac{3x}{y}\right] y\,dx$

$y\,dy = 3x\,dx$

$\int y\,dy = \int 3x\,dx$

$\frac{y^2}{2} = \frac{3}{2}x^2 + C_1$ implicit soln

$y^2 = 3x^2 + \underbrace{2C_1}_{\equiv C_1'}$

$y = \pm\sqrt{3x^2+C_1'}$ two solns

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missing step! $\int \frac{dy}{y} = \int 3x\,dx$

$\ln|y| + C_y = \frac{3}{2}x^2 + C_x$

$\ln|y| = \frac{3}{2}x^2 + \underbrace{C_x - C_y}_{\equiv C_1}$ only difference of two constants of integration appears, rename it as the new arbitrary constant gravitational force on mass m at height x above Earth surface:

$$F = -\frac{GMm}{(x+R)^2} = -\frac{GM}{R^2}\cdot\frac{R^2 m}{(x+R)^2} = -\frac{mgR^2}{(x+R)^2} = m\underbrace{\frac{dv}{dt}}_{\text{acceleration}} = m\frac{d^2x}{dt^2}$$

second order DE for $x(t)$

downward

$(x+R)$: distance from center of Earth, where x = height above Earth surface

$\frac{GM}{R^2} = g$ = gravity acceleration at Earth surface

$v = \frac{dx}{dt}$

If we change independent variable from t to x:

$$v = v(x(t)) \longrightarrow \frac{dv}{dt} = \frac{dv}{dx}\underbrace{\frac{dx}{dt}}_{v} = v\frac{dv}{dx} \longrightarrow m v\frac{dv}{dx} = \frac{mgR^2}{(x+R)^2}$$

This gives us velocity as a function of height.

now first order DE for $v(x)$

$\boxed{\text{solution}}$

$$v\frac{dv}{dx} = \frac{-gR^2}{(x+R)^2} \rightarrow \quad v\,dv = -gR^2(x+R)^{-2}dx$$

$$\int v\,dv = -\int gR^2(x+R)^{-2}dx$$

$$m\left[\frac{v^2}{2} = -gR^2\frac{(x+R)^{-1}}{-1} + C\right]$$

$$\frac{1}{2}mv^2 = \frac{mgR^2}{x+R} + mC \quad \leftarrow \text{very useful implicit solution}$$

"conservation of energy":

$$\underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}} \underbrace{-\frac{mgR^2}{x+R}}_{\text{potential energy}} = \underbrace{mC}_{\text{total energy is constant}} \equiv E = \frac{1}{2}mv_0^2 - \frac{mgR^2}{R}$$

energy value at $x=0$

$$= \frac{1}{2}m(0)^2 - \frac{mgR^2}{h+R}$$

if velocity slows to zero at maximum height h

equate 2 RHSs

$$\text{so } -\frac{mgR^2}{h+R} = -\frac{mgR^2}{R} + \frac{1}{2}mv_0^2$$

$$\frac{1}{2}v_0^2 = \frac{gR^2}{R} - \frac{gR^2}{h+R} = gR^2\left(\frac{1}{R} - \frac{1}{h+R}\right) = gR^2\left(\frac{h+R-R}{R(h+R)}\right) = \frac{gRh}{h+R}$$

$$v_0^2 = \frac{2ghR}{(h+R)} \qquad v_0 = \sqrt{\frac{2Rgh}{(h+R)}} = \text{initial speed needed to reach height } h.$$

$$v_{escape} = \lim_{h\to\infty}\sqrt{\frac{2Rgh}{h+R}} = \lim_{h\to\infty}\sqrt{\frac{2Rg}{1+R/h}} = \sqrt{2gR}$$

$g = 32\,ft/s^2$, $R = 3960$ mi $= 3960 \cdot 5280$ ft

$v_{escape} = 36{,}581$ ft/s $= 24942$ mph

This is the initial speed needed to escape the Earth's gravitational field.

Notice what a small part the integration step played in this discussion. "Solving a DE" is a small part of learning how to use DEs and their solns to understand physical systems. There is a lot to learn besides producing soln formulas!