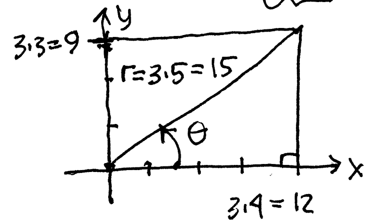
MAT2500-03/04 14S Final Exam Answers (1)

1. $(x,y,z) = (\underbrace{12,9}_{},8)$

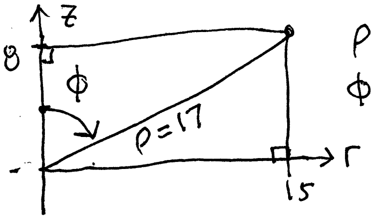


$\theta = \arctan\frac{y}{x} = \arctan\frac{9}{12}$

$= \arctan 3/4$

$r = \sqrt{12^2+9^2} = \sqrt{3^2 4^2+3^2 3^2}$

$= 3\sqrt{4^2+3^2} = 3\sqrt{5^2} = 15$



$\rho = \sqrt{8^2+15^2} = 17$

$\phi = \arctan\frac{15}{8} = \arccos\frac{8}{17}$

$= \arcsin\frac{15}{17}$

a) cylindrical: $(r,\theta,z) = (15, \arctan\frac{3}{4}, 8)$

$\approx 36.9°$

b) spherical: $(\rho,\phi,\theta) = (17, \arctan\frac{15}{8}, \arctan 3/4)$

$\approx 61.9°$

θ is clearly between 30° & 45° ✓

ϕ is about 60° ✓

so yes, the angles look right.

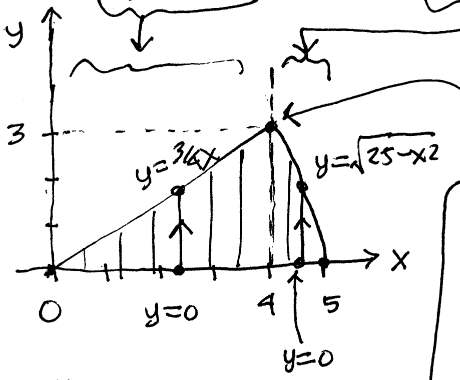
2. a) $\displaystyle\int_{x=0}^{x=4}\int_{y=0}^{y=\frac{3}{4}x} f\,dy\,dx + \int_{x=4}^{x=5}\int_{y=0}^{y=\sqrt{25-x^2}} f\,dy\,dx$

$y \geq 0$

$y^2 = 25-x^2$

$x^2+y^2 = 25 = 5^2$

circle radius 5
center (0,0)



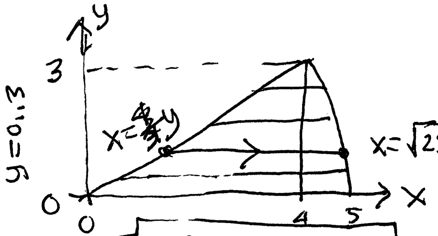
intersection:

$\frac{3}{4}x = \sqrt{25-x^2}$

$\frac{9}{16}x^2 = 25-x^2$

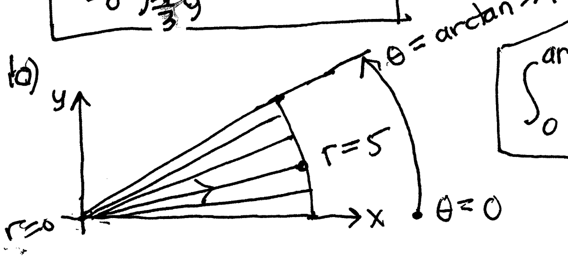
$\frac{25}{16}x^2 = 25$

$x^2 = 16,\ x = 4$



$$\boxed{\int_0^3\int_{\frac{4}{3}y}^{\sqrt{25-y^2}} f\,dx\,dy}$$

b)



$$\boxed{\int_0^{\arctan\frac{3}{4}}\int_0^5 f\,\underbrace{r\,dr\,d\theta}_{dA}}$$

c) $\displaystyle\int_0^4\underbrace{\int_0^{\frac{3}{4}x} dy}_{\frac{3}{4}x}\,dx = \frac{3}{4}\frac{x^2}{2}\Big|_0^4 = 6$

$\displaystyle\int_4^5\int_0^{\sqrt{25-x^2}} dy\,dx \overset{\text{Maple}}{=} -6 + \frac{25}{4}\pi - \frac{25}{2}\arcsin\frac{4}{5}$

sum $= \boxed{\frac{25}{4}\pi - \frac{25}{2}\arcsin\frac{4}{5} \approx 8.0438} = A$

$\displaystyle\int_0^3\underbrace{\int_{\frac{4}{3}y}^{\sqrt{25-y^2}} dx}_{\sqrt{25-y^2}-\frac{4}{3}y}\,dy \overset{\text{Maple}}{=} \boxed{\frac{25}{2}\arcsin\left(\frac{3}{5}\right) \approx 8.0438} = A$

$\displaystyle\int_0^{\arctan\frac{3}{4}}\underbrace{\int_0^5 r\,dr}_{\frac{r^2}{2}\Big|_0^5 = \frac{25}{2}}\,d\theta = \frac{25}{2}\theta\Big|_0^{\arctan 3/4} = A$

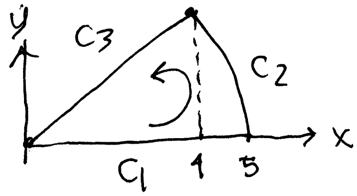
$= \boxed{\frac{25}{2}\arctan\left(\frac{3}{4}\right) \approx 8.0438}$

d) $\vec{F} = \langle x-y, x+y\rangle = \langle F_1, F_2\rangle$

$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(x+y) = 1 \qquad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1$

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1-(-1) = 2$

$\displaystyle\iint_R \frac{\partial F}{\partial x} - \frac{\partial F_1}{\partial y}\,dA = \iint_R 2\,dA = 2\,\text{Area}(R) = 2A$



C_1: $x = t = 0..5$, $y = 0$, $\vec{r} = \langle t, 0\rangle$
$\vec{r}\,' = \langle 1, 0\rangle$

$\vec{F}(\vec{r}(t)) = \langle t-0, t+0\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \langle t,t\rangle\cdot\langle 1,0\rangle = t$

$\int_{C_1} F\cdot d\vec{r} = \int_0^5 t\,dt = \frac{t^2}{2}\Big|_0^5 = \frac{25}{2}$

C_3: $x = t$, $y = \frac{3}{4}t$, $\vec{r} = \langle t, \frac{3}{4}t\rangle$ $t = 4..0$
$\vec{r}\,' = \langle 1, 3/4\rangle$

$\vec{F}(\vec{r}(t)) = \langle t - \frac{3}{4}t, t + \frac{3}{4}t\rangle = \langle -\frac{t}{4}, \frac{7}{4}t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \frac{t}{4}\langle -1,7\rangle\cdot t\langle 1,\frac{3}{4}\rangle = \frac{25}{16}t$

$\int_{C_3}\vec{F}\cdot d\vec{r} = \int_4^0 \frac{25}{16}t\,dt = -\frac{25}{16}\frac{t^2}{2}\Big|_0^4 = -\frac{25}{2}$

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2. d) continued $r=5,\ \theta=t=0..\arctan 3/4$

C_2: $\vec{r}=\langle x,y\rangle=\langle 5\cos t, 5\sin t\rangle$

$\vec{r}\,'=5\langle \sin t, \cos t\rangle$

$\vec{F}(\vec{r}(t))=\langle 5c-5s,\ 5c+5s\rangle$

$=5\langle c-s, c+s\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)=5\langle c-s,c+s\rangle\cdot 5\langle -s,c\rangle$

$=25(-cs+s^2+c^2+sc)=25$

$\int_{C_2}\vec{F}\cdot d\vec{r}=\int_0^{\arctan 3/4}25\,dt=25\arctan 3/4$

$=2A$ ✓

3. a) $\underbrace{x^2+y^2+z^2}_{\rho^2}=17\to\rho=\sqrt{17}$, $\underbrace{x^2+y^2}_{r^2}=16\to r=4$

height

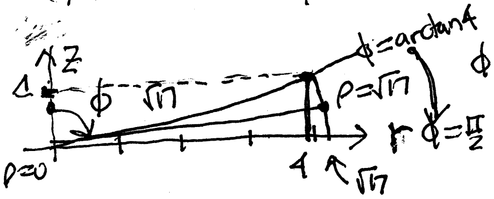
$\boxed{b)\ z=1},\ \boxed{r=4}$

Intersection: $\underbrace{x^2+y^2}_{x^2+y^2}+z^2=17$

$=16$

$-$: $z^2=1,\ z=\pm1\to \underline{1}$ for $z\geq0$.

$(r,z)=(4,1)\to\rho=\sqrt{4^2+1^2}=\sqrt{17}$ ✓

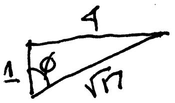


$\phi=\arctan 4..\frac{\pi}{2}$

c) $V=\int_0^{2\pi}\int_{\arctan 4}^{\arctan 4}\int_0^{\sqrt{17}}1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ factors

$=\underbrace{\int_0^{2\pi}d\theta}_{2\pi}\ \underbrace{\int_0^{\arctan 4}\sin\phi\,d\phi}_{-\cos\phi\big|_{\arctan 4}^{\pi/2}}\ \underbrace{\int_0^{\sqrt{17}}\rho^2\,d\rho}_{\frac{\rho^3}{3}\big|_0^{\sqrt{17}}=\frac{17\sqrt{17}}{3}}$

$=0+\cos(\arctan 4)$



$\phi=\arccos\frac{1}{\sqrt{17}}=\arcsin\frac{4}{\sqrt{17}}$ ($\approx 76.0°$)

$\cos\phi=\frac{1}{\sqrt{17}}$

$V=2\pi\left(0-\frac{1}{\sqrt{17}}\right)\frac{17\sqrt{17}}{3}=\frac{17\cdot2\pi}{3}=\boxed{\frac{34\pi}{3}\approx35.6047}$

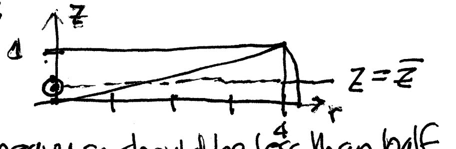
$V\bar{z}=\int_0^{2\pi}\int_{\arctan 4}^{\pi/2}\int_0^{\sqrt{17}}(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$=\underbrace{\int_0^{2\pi}d\theta}_{2\pi}\ \int_{\arctan 4}^{\pi/2}\sin\phi\cos\phi\,d\phi\ \underbrace{\int_0^{\sqrt{17}}\rho^3\,d\rho}_{\rho^4/4\big|_0^{\sqrt{17}}=\frac{17^2}{4}}$

$\frac{\sin^2\phi}{2}\Big|_{\arctan 4}^{\pi/2}$

$=2\pi\,\frac{1}{2}\left(1-\left(\frac{4}{\sqrt{17}}\right)^2\right)\frac{17^2}{4}=\frac{2\pi}{8}\left(1-\frac{16}{17}\right)17^2=\frac{\pi}{4}\cdot17=\boxed{\frac{17}{4}\pi\approx13.3518}$

③ $\bar{z}=\frac{V\bar{z}}{V}=\frac{\frac{17}{4}\pi}{\frac{34\pi}{3}}=\boxed{\frac{3}{8}=0.375}$



bottom heavy so should be less than half height ✓

so makes sense.

④ a) $\vec{F}=\langle x,y,z\rangle=\vec{\nabla}f$?

$\text{curl}\,\vec{F}=\vec{\nabla}\times\vec{F}=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=\left\langle\frac{\partial z}{\partial y}-\frac{\partial x}{\partial z},\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x},\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right\rangle$

$=\langle0,0,0\rangle$

b) $\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle=\langle x,y,z\rangle$

$\int\left[\frac{\partial f}{\partial x}=x\right]dx\to f=\frac{x^2}{2}+C(y,z)\to\frac{\partial f}{\partial y}=0+\frac{\partial C}{\partial y}(y,z)$

$\frac{\partial f}{\partial y}=y\to\frac{\partial C}{\partial y}(y,z)=y\to C(y,z)=\frac{y^2}{2}+C(z)$

$\frac{\partial f}{\partial z}=z$ $\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}\left(\frac{x^2}{2}+\frac{y^2}{2}+C(z)\right)=C'(z)$

$=z$

$C(z)=\frac{z^2}{2}+k$ so $\boxed{f=\frac{1}{2}(x^2+y^2+z^2)+k}$

c) $\int_C\vec{F}\cdot d\vec{r}=f(2,0,4)-f(0,0,0)$

$=\frac{1}{2}(2^2+0^2+4^2)+k-k=\boxed{10}$

d) $\vec{r}=\langle0,0,0\rangle+t(\langle2,0,4\rangle-\langle0,0,0\rangle)=\langle2t,0,4t\rangle,\ t=0..1$

$\vec{r}\,'=\langle2,0,4\rangle$

$\vec{F}=\vec{r}=\langle2t,0,4t\rangle$

$\vec{F}\cdot r'=\langle2t,0,4t\rangle\cdot\langle2,0,4\rangle=t(4+16)=20t$

$\int_{C_2}\vec{F}\cdot d\vec{r}=\int_0^1 20t\,dt=\frac{20t^2}{2}\Big|_0^1=\boxed{10}$

e) $\vec{r}=\langle x,y,z\rangle=\langle 2c^2, 2cs, 4c^2\rangle$

$r^2=x^2+y^2=4c^4+4c^2s^2=4c^2(c^2+s^2)=4c^2$ z

so $\boxed{z=r^2}$ ✓ $\vec{r}(\frac{\pi}{2})=\langle0,0,0\rangle$, $\vec{r}(\pi)=\langle2,0,4\rangle$ ✓

f) $\vec{F}\cdot\vec{r}\,'=\vec{r}\cdot\vec{r}\,'=2c\langle c,s,2c\rangle\cdot2\langle-2cs,c^2-s^2,-4cs\rangle$

$=4c(-\cancel{2c^2s}+\underset{0}{c^2s}-s^3-8c^2s)=-4s^3(+c)+36c^3(-s)$

$\int_{C_3}F\cdot dr=\int_{\frac{\pi}{2}}^{\pi}-4s^3\,ds+\int_{\frac{\pi}{2}}^{\pi}36c^3\,dc=-\sin^4t+9\cos^4t\Big|_{\pi/2}^{\pi}$

$=(-0+1)+9(1-0)=\boxed{10}$ ✓ they all must agree since the result is path independent. they do.