MAT 2500-03/04 14S Take home test 3 Answers

① a $\int_2^4 \int_{-\sqrt{4y-y^2}}^{0} x \, dx \, dy = \int_2^4 \frac{x^2}{2} \Big|_{x=-\sqrt{4y-y^2}}^{x=0} dy$

$= \int_2^4 \underbrace{-\frac{1}{2}(4y-y^2)}_{-2y+\frac{1}{2}y^2} dy = \frac{1}{6}y^3 - y^2 \Big|_2^4$

$= \frac{1}{6}(4^3-2^3) - (4^2-2^2) = \frac{\overset{28}{\cancel{56}}}{\cancel{6}\,3} - 12 = \frac{28-36}{3} = \boxed{-\frac{8}{3}}$

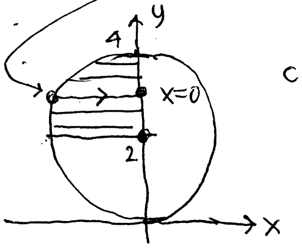
b) $\int_{y=2}^{y=4} \int_{x=-\sqrt{4y-y^2}}^{x=0} \cancel{x} \, dx \, dy$

left half of circle:

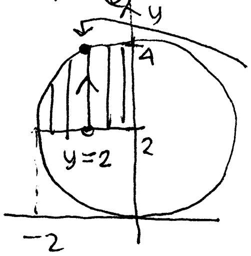
$x^2 = 4y - y^2$

$x^2 + y^2 - 4y = 0$

$x^2 + (y-2)^2 = 4$



c)



$y = 2 + \sqrt{4-x^2}$

c) $y^2 - 4y + x^2 = 0$

$y = \frac{4 \pm \sqrt{16-4x^2}}{2} = 2 \pm \sqrt{4-x^2}$ upper half

$\int_{-2}^{0} \underbrace{\int_2^{2+\sqrt{4-x^2}} x \, dy}\, dx$

$= \underline{xy} \Big|_{y=2}^{y=2+\sqrt{4-x^2}} = x(2+\sqrt{4-x^2}) - 2x$

$= x\sqrt{4-x^2}$

$= \int_{-2}^{0} [\underbrace{(4-x^2)^{1/2}}_{u=-x^2} \underbrace{x]dx}_{\to -\frac{dy}{2}},$

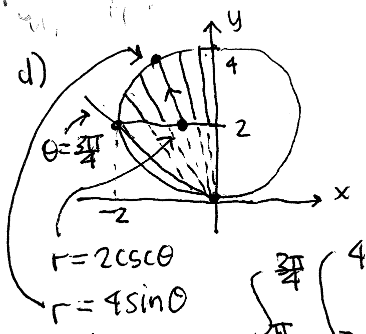
$du = -2x dx$

$\int u^{1/2}(-\frac{dy}{2}) = -\frac{1}{2}\frac{u^{3/2}}{3/2}$

$= -\frac{1}{3}(4-x^2)^{3/2} \Big|_{-2}^{0} = \boxed{-\frac{8}{3}}$

polar coordinates emanate from the origin!

d)



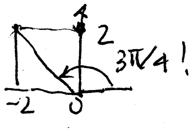
$y = 2 \to r\sin\theta = 2$

$r = 2\csc\theta$

$x^2 + y^2 = 4y$

$r^2 = 4r\sin\theta$

$r = 4\sin\theta$



$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_{\underset{\csc}{2\csc\theta}}^{4\sin\theta} \underbrace{(r\cos\theta)}_{x} \underbrace{r\,dr\,d\theta}_{dA}$

d) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \underbrace{\int_{2\csc\theta}^{4\sin\theta} r^2 \cos\theta \, dr}\, d\theta$

$\frac{r^3}{3}\cos\theta \Big|_{r=2\csc\theta}^{r=4\sin\theta} = \frac{1}{3}(64\sin^3\theta - 8\csc^3\theta)\overset{\cos\theta}{\vee}$

$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{1}{3}(64\sin^3\theta - 8\sin^{-3}\theta) \underbrace{\cos\theta \, d\theta}_{du}$

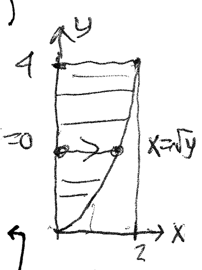
$\frac{1}{3}(64u^3 - 8u^{-3})$

$\frac{1}{3}(64\frac{u^4}{4} - 8\frac{u^{-2}}{-2}) = \frac{1}{3}(16\sin^4\theta + 4\csc^2\theta)$

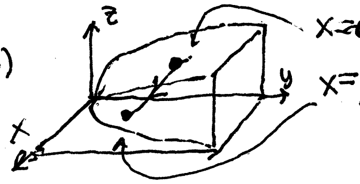
$= \frac{1}{3}(16\sin^4\theta + 4\csc^2\theta) \Big|_{\pi/2}^{3\pi/4}$

$= \frac{1}{3}(16(\frac{1}{2})^2 + 4(2)) - \frac{1}{3}(16+4)$

$= \frac{1}{3}(12-20) = \boxed{-\frac{8}{3}}$ ☺



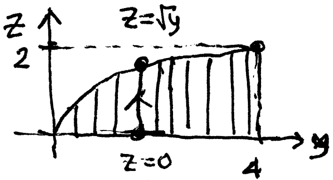
② a)



X=0 backwall

$x = y^{1/2}$ frontwall

$y = x^2$ → $x = y^{1/2}$



oops, TV watching distraction...

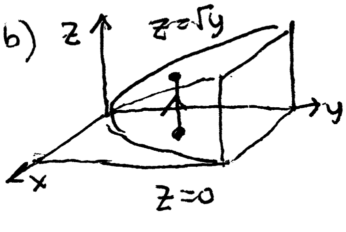
$V = \int_0^4 \int_0^{\sqrt{y}} \underbrace{\int_0^{y^{1/2}} 1 \, dx}\, dz \, dy$

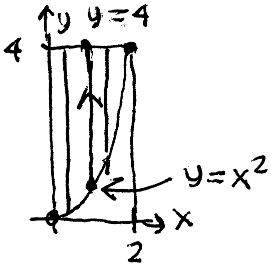
$x \Big|_{x=0}^{x=y^{1/2}} = y^{1/2}$

$\int_0^{\sqrt{y}} y^{1/2} dz = y^{1/2} z \Big|_{z=0}^{z=\sqrt{y}} = y^{1/2}y^{1/2} = y$

$= \int_0^4 y \, dy = \frac{y^2}{2} \Big|_0^4 = \frac{2}{7}~~(2^2)^{7/2} - \frac{2^8}{7}~~ = \frac{16}{2} = \boxed{8}$

b)





$\int_0^2 \int_{x^2}^4 \underbrace{\int_0^{\sqrt{y}} 1 \, dz}\, dy \, dx$

$z \Big|_{z=0}^{z=\sqrt{y}} = \sqrt{y}$

$\frac{y^{3/2}}{3/2} \Big|_{y=x^2}^{y=4} = \frac{2}{3}(4^{3/2} - (x^2)^{3/2}) = \frac{16}{3} - \frac{2x^3}{3}$

$= \frac{16}{3}x - \frac{x^4}{\cancel{6}} \Big|_0^2 = \frac{16}{3}(2) - \frac{16}{\cancel{6}} = \frac{32-8}{3} = \frac{24}{3} = \boxed{8}$

c) Yup! Maple concurs.

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③ a) circle center (r_0, z_0), radius a:

$$(r-r_0)^2+(z-z_0)^2=a^2$$

$C(0,0), a=8$: $\boxed{r^2+z^2=64}$

$C(2,0), a=5$: $\boxed{(r-5)^2+z^2=25}$

b) $r^2-10r+25+z^2=25 \rightarrow r^2+z^2=10r$

$-(r^2 + z^2=64)$

$-10r+\cancel{25}=\cancel{25}-64=$

$r=\frac{64}{10}=\frac{32}{5}$

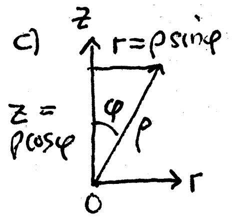
$r^2+z^2=64 \rightarrow z=\sqrt{64-r^2}=\sqrt{64-\left(\frac{32}{5}\right)^2}$

$=\sqrt{64\left(1-\frac{16}{25}\right)}=8\sqrt{\frac{9}{25}}=\frac{3}{5}\cdot 8=\frac{24}{5}$

$\boxed{(r,z)=\left(\frac{32}{5},\frac{24}{5}\right)\equiv(R,Z)}=(6.4, 4.8)$ agrees with plot!

$\frac{z}{r}=\frac{24}{32}=\frac{3}{4}$ $\boxed{z=\frac{3}{4}r}$

c)



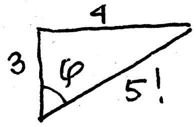
$\rho^2=r^2+z^2=64$

$\hookrightarrow \boxed{\rho=8}$

$\rho^2=r^2+z^2=10(\rho\sin\varphi)$

$\boxed{\rho=10\sin\varphi}$

d) $\tan\varphi=\frac{r}{z}=\frac{4}{3} \rightarrow \boxed{\varphi=\arctan\frac{4}{3}} \equiv \Phi$



$\rho=10\sin\varphi$

$=10\left(\frac{4}{5}\right)=8 \equiv P$

we already knew that

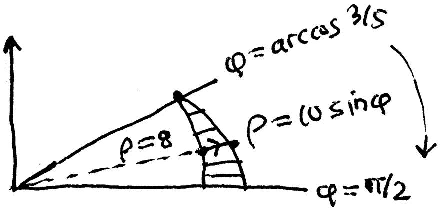
$\therefore \sin\varphi=\frac{4}{5}$

$\cos\varphi=\frac{3}{5}$

$\boxed{\varphi=\arcsin\frac{4}{5}=\arccos\frac{3}{5}} \equiv \Phi$

equivalent

e) $\langle V, M_{xy}\rangle$:



$$\langle V, M_{xy}\rangle=\underbrace{\int_0^{2\pi}}_{2\pi}\int_{\arccos 3/5}^{\pi/2}\int_8^{10\sin\varphi}\langle 1, \underbrace{\rho\cos\varphi}_{z}\rangle\rho^2\sin\phi\, d\rho\, d\varphi\, d\theta$$

f) $$\int_8^{10\sin\varphi}\langle\rho^2, \rho^3\cos\phi\rangle\sin\phi\, d\rho$$

$$=\left\langle\frac{\rho^3}{3}, \frac{\rho^4}{4}\cos\varphi\right\rangle\sin\phi\ \Big|_{\rho=8}^{\rho=10\sin\varphi}$$

$$=\left\langle\frac{1000}{3}\sin^4\phi-\frac{512}{3}\sin\varphi, \frac{10^4}{4}\sin^5\varphi\cos\varphi-\frac{8^4}{4}\sin\varphi\right\rangle$$

e) $$V=2\pi\int_{\arccos 3/5}^{\frac{\pi}{2}}\left(\frac{1000}{3}\sin^4\phi-\frac{512}{3}\sin\phi\right)d\phi$$

$$=2\pi\left\{\frac{1000}{3}\left(-\frac{1}{4}\sin^3\phi\cos\phi-\frac{3}{8}\cos\phi\sin\phi+\frac{3}{8}\phi\right)+\frac{512}{3}\cos\phi\right\}\Big|$$

$$=2\pi\left\{125\left(\frac{\pi}{2}\right)+\underbrace{\frac{250}{3}\left(\frac{4}{5}\right)^3\left(\frac{3}{5}\right)+125\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)-\frac{512}{3}\left(\frac{3}{5}\right)}_{\doteq -84/5}-125\arctan\frac{4}{3}\right.$$

$$=2\pi\left\{\frac{125\pi}{2}-\frac{84}{5}-125\arctan\frac{4}{3}\right\}\approx\boxed{399.8471}$$

$$M_{xy}=2\pi\int_{\arccos 3/5}^{\frac{\pi}{2}}\left(\frac{10^4}{4}\underbrace{\sin^5\phi}_{u^5}\underbrace{\cos\varphi}_{\frac{du}{d\phi}}-\frac{8^4}{4}\underbrace{\sin\phi}_{(-\cos^2\phi)}\right)d\phi$$

$\frac{u^6}{6}=\frac{\cos^6\phi}{6}$

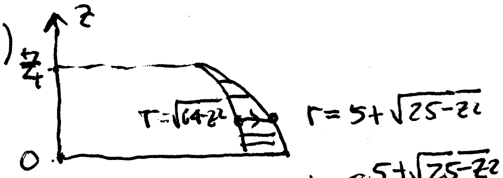
$$=2\pi\left\{\underbrace{\frac{10^4}{24}}_{1250/3}\sin^6\phi+\underbrace{\frac{8^4}{4}}_{512}\cos^2\phi\right\}\Big|_{\arccos 3/5}^{\pi/2}$$

$$=2\pi\left\{\underbrace{\frac{1250}{3}\left(1-\left(\frac{4}{5}\right)^6\right)-512\left(\frac{3}{5}\right)^2}_{\frac{3078}{25}}\right\}=\frac{6156\pi}{25}$$

$$\approx\boxed{773.5858}$$

$$\bar{z}=\frac{M_{xy}}{V}\approx 1.9347\approx\boxed{1.93}$$

g)



$$\langle V, M_{xy}\rangle=\int_0^{2\pi}\int_0^{Z}\int_{\sqrt{64-z^2}}^{5+\sqrt{25-z^2}}\langle 1, z\rangle\, r\, dr\, dz\, d\theta$$

h)

$$=2\pi\left\{\int_0^{Z}\underbrace{\int_{\sqrt{64-z^2}}^{5+\sqrt{25-z^2}}\langle r, rz\rangle\, dr}_{\left\langle\frac{r^2}{2}, \frac{r^2}{2}z\right\rangle\Big|_{r=\sqrt{64-z^2}}^{r=5+\sqrt{25-z^2}}}dz\right\}$$

$$\left[(5+\sqrt{25-z^2})^2-(64-z^2)\right]\frac{\langle 1, z\rangle}{2}=-7+5\sqrt{25-z^2}\langle 1, z\rangle$$

$$=2\pi\int_0^{Z}\langle -7+5\sqrt{25-z^2}, -7z+5z\sqrt{25-z^2}\rangle dz$$

$$=2\pi\left\{\left\langle -7z+\frac{5}{2}z\sqrt{25-z^2}+\frac{125}{2}\arcsin\left(\frac{z}{5}\right), -\frac{7}{2}z^2-\frac{5}{3}(25-z^2)^{3/2}\right\rangle\right\}\Big|_0^{24/5}$$

$$=2\pi\left\{\left\langle\underbrace{-\frac{168}{5}+\frac{84}{5}}_{=-84/5}+\arcsin\left(\frac{24}{25}\right), -\frac{7}{2}\left(\frac{24}{5}\right)^2-\left(\frac{7}{5}\right)^3\frac{5}{3}+\frac{5^4}{3}\right\rangle\right.$$

MAT2500-03/04 F15 Takehome Test 3 Answers (3)

③ h) $V = 2\pi\left(\arcsin\left(\frac{24}{25}\right) - \frac{84}{5}\right) \approx \boxed{399.8471}$

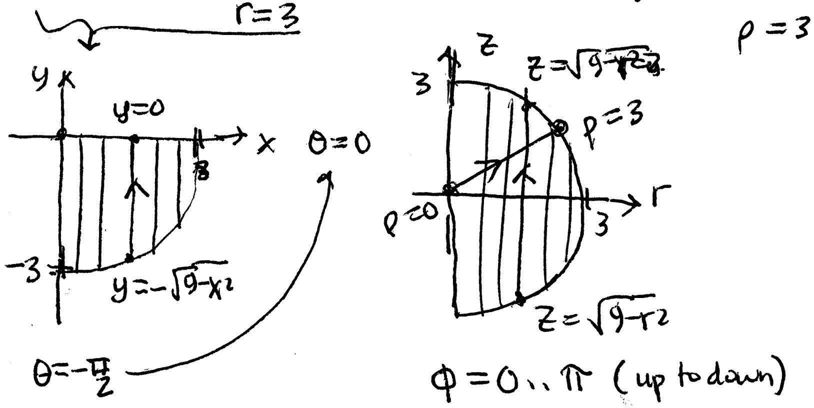
$M_{xy} = 2\pi\left(\frac{3078}{25}\right) \approx \boxed{773.5858}$ ✓

same $\bar{z}$. no need to give details for integration here.

i) This is clearly bottom heavy so the centroid must be below the half-height point somewhat and it is. See plot.

④ a) $$\underbrace{\int_{x=0}^{x=3}\int_{y=-\sqrt{9-x^2}}^{y=0}}_{x^2+y^2=9,\ r=3}\ \underbrace{\int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}}}_{z^2=9-x^2-y^2 \rightarrow x^2+y^2+z^2=9=\rho^2,\ \rho=3} \underbrace{x^2+y^2+z^2}_{\rho^2}\ \underbrace{dz\,dy\,dx}_{dV=\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

(b) $\underset{\text{MAPLE}}{=}$ $\boxed{\frac{243\pi}{5}}$



$\phi = 0..\pi$ (up to down)

$$\boxed{\int_{-\frac{\pi}{2}}^{0}\int_{0}^{\pi}\int_{0}^{3} \rho^2\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta} \underset{\text{Maple}}{\overset{b)}{=}} \boxed{\frac{243\pi}{5}}$$ they agree

c) $= \int_{-\frac{\pi}{2}}^{0} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^3 \rho^4\, d\rho$

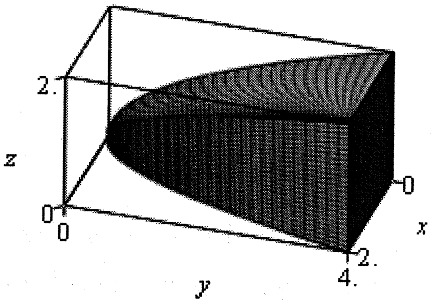
$= \theta\Big|_{-\pi/2}^{0}\ (-\cos\phi)\Big|_0^{\pi}\ \frac{\rho^5}{5}\Big|_0^3$

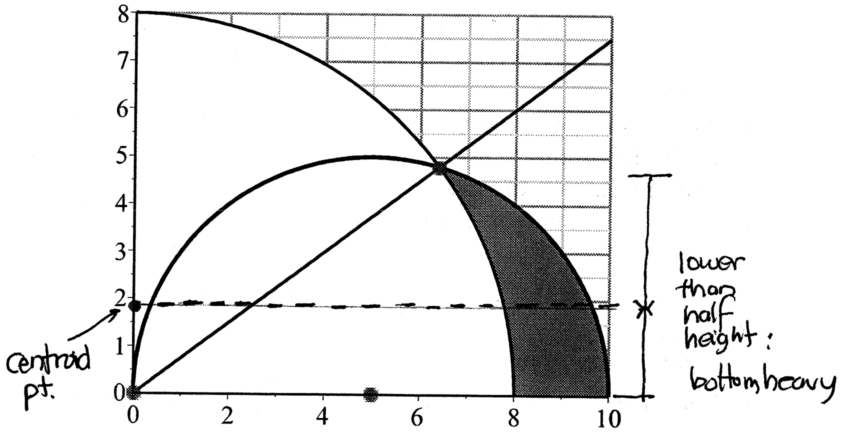
$= \frac{\pi}{2}\,(1+1)\cdot\frac{3^5}{5} = \frac{3^5}{5}\pi = \boxed{\frac{243\pi}{5}} \approx \boxed{152.68}$ agrees!

4. **Cartesian to spherical coordinate integral conversion.**

a) Use spherical coordinates to represent $\int_0^3 \int_{-\sqrt{9-x^2}}^{0} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} x^2+y^2+z^2 \,dz\,dy\,dx$ as a triple integral, supporting your limits of integration with labeled diagrams as above.
b) Evaluate this new integral exactly with technology and compare its value with the original triple integral in Cartesian coordinates evaluated exactly and numerically with technology. Do they agree?
c) Evaluate the spherical coordinate integral exactly by hand step by step. Does it agree with your previous results?

Note that the origin is in the back left corner of the left diagram.





▶ solution (on-line)

No collaboration. You may only talk to bob. See test rules on-line. Read short rules above. Print out and attach any Maple supporting work you do, hand annotating if necessary with problem number and part etc, whatever is necessary for clarification.

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge if it applies and hand in stapled to your answer sheets as the cover page, with the Lastname, FirstName side face up:
"During this examination, all work has been my own. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants.
"

Signature: Date: