MAT2705-01/02 14F Final Exam Answers(1)

① a) $x_1 = -\frac{48}{7}\cos 3t + 4\cos 2t + \frac{20}{7}\cos 4t - 2\sin 2t + \sin 4t$

$x_2 = -\frac{3}{7}\cos 3t + \cos 2t - \frac{10}{7}\cos 4t - \frac{1}{2}\sin 2t - \frac{1}{2}\sin 4t$

b) $\omega_1 = 2,\ \omega_2 = 4,\ \omega = 3$

$T_1 = \frac{2\pi}{2} = \pi,\ T_2 = \frac{2\pi}{4} = \frac{\pi}{2},\ T = \frac{2\pi}{3}$

2 cycles 4 cycles 3 cycles

c) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \begin{bmatrix} -8 & 16 \\ 2 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 15\cos 3t \end{bmatrix}$

$\vec{x}'' = A\vec{x} + \vec{F} : A = \begin{bmatrix} -8 & 8 \\ 1 & -12 \end{bmatrix},\ \vec{F} = \begin{bmatrix} 0 \\ 15\cos 3t \end{bmatrix}$

d) $0 = |A - \lambda I| = \begin{vmatrix} -8-\lambda & 16 \\ 2 & -12-\lambda \end{vmatrix} = (\lambda+8)(\lambda+12) - 32$

$= \lambda^2 + 20\lambda + 96 - 32 = \lambda^2 + 20\lambda + 64$

$= (\lambda+4)(\lambda+16) \rightarrow \lambda = -4, -16$

$\lambda = -4$: $A + 4I = \begin{bmatrix} -8+4 & 16 \\ 2 & -12+4 \end{bmatrix} = \begin{bmatrix} -4 & 16 \\ 2 & -8 \end{bmatrix}$

$\xrightarrow{\text{rref}}$ L F $\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2 = t$, $x_1 = 4x_2 = 4t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix} \rightarrow \vec{b}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, e) $\boxed{m_1 = 1/4}$

$\lambda = -16$: $A + 16I = \begin{bmatrix} -8+16 & 16 \\ 2 & -12+16 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 2 & 4 \end{bmatrix}$

$\xrightarrow{\text{rref}}$ L F $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2 = t$, $x_1 = -2x_2 = -2t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, e) $\boxed{m_2 = -1/2}$

$B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

$B^{-1} = \frac{1}{4+2}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

$B^{-1}AB = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix} = A_B$

f) $B^{-1}F = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 15\cos 3t \end{bmatrix} = \begin{bmatrix} 5\cos 3t \\ 10\cos 3t \end{bmatrix}$

$B^{-1}\vec{x}'(0) = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \vec{y}'(0)$

g) $\boxed{\vec{x}'(0) = -\vec{b}_1 - 2\vec{b}_2}$

h) $B^{-1}[\vec{x}'' = A\vec{x} + \vec{F}]$

$B^{-1}[(B\vec{y})'' = A(B\vec{y}) + \vec{F}]$

$\vec{y}'' = A_B\vec{y} + B^{-1}F$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 5\cos 3t \\ 10\cos 3t \end{bmatrix} = \begin{bmatrix} -4y_1 + 5\cos 3t \\ -16y_2 + 10\cos 3t \end{bmatrix}$

h) $y_1'' = -4y_1 + 5\cos 3t,\quad y_2'' = -16y_2 + 10\cos 3t$

$\boxed{y_1'' + 4y_1 = 5\cos 3t \qquad y_2'' + 16y_2 = 10\cos 3t}$

$y_{1h} = c_1\cos 2t + c_2\sin 2t \qquad y_{2h} = c_3\cos 4t + c_4\sin 4t$

$4\,[y_{1p} = c_5\cos 3t + c_6\sin 3t] \qquad 16\,[y_{2p} = c_7\cos 3t + c_8\sin 3t]$

$1\,[y_{1p}'' = -9c_5\cos 3t - 9c_6\sin 3t] \qquad 1\,[y_{2p}'' = -9c_7\cos 3t - 9c_8\sin 3t]$

$y_{1p}'' + 4y_p = (4-9)c_5\cos 3t + (4-9)c_6\sin 3t = 5\cos 3t$

$-5c_5 = 5 \quad -5c_6 = 0$

$\hookrightarrow c_5 = -1 \quad \hookrightarrow c_6 = 0$

$y_{1p} = -\cos 3t$

$y_{2p}'' + 16y_p = (16-9)c_7\cos 3t + (16-9)c_8\sin 3t = 10\cos 3t$

$7c_7 = 10 \quad 7c_8 = 0$

$\hookrightarrow c_7 = 10/7 \quad \hookrightarrow c_8 = 0$

$y_{2p} = \frac{10}{7}\cos 3t$

$$\boxed{\begin{aligned} y_1 &= \underbrace{c_1\cos 2t + c_2\sin 2t}_{\text{homogeneous part}} \underbrace{- \cos 3t}_{\text{particular part}} \\ y_2 &= c_3\cos 4t + c_4\sin 4t + \tfrac{10}{7}\cos 3t \end{aligned}}$$

i) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B\begin{bmatrix} c_1\cos 2t + c_2\sin 2t - \cos 3t \\ c_3\cos 4t + c_4\sin 4t + \frac{10}{7}\cos 3t \end{bmatrix}$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = B\begin{bmatrix} -2c_1\sin 2t + 2c_2\cos 2t + 3\sin 3t \\ -4c_3\sin 4t + 4c_4\cos 4t - \frac{21}{7}\sin 3t \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B\begin{bmatrix} c_1 - 1 \\ c_3 + 10/7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} c_1 - 1 \\ c_3 + 10/7 \end{bmatrix} = B^{-1}\vec{0} = \vec{0}$

$c_1 = 1,\ c_3 = -10/7$

$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = B\begin{bmatrix} 2c_2 \\ 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix},\ \begin{bmatrix} 2c_2 \\ 4c_4 \end{bmatrix} = B^{-1}\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ $2c_2 = -1,\ c_2 = -1/2$; $4c_4 = -2,\ c_4 = -1/2$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos 2t - \frac{1}{2}\sin 2t - \cos 3t \\ -\frac{10}{7}\cos 4t - \frac{1}{2}\sin 4t - \frac{10}{7}\cos 3t \end{bmatrix}$

sign miscopying sloppiness!

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \cos 2t - \frac{1}{2}\sin 2t - \cos 3t \\ -\frac{10}{7}\cos 4t - \frac{1}{2}\sin 4t + \frac{10}{7}\cos 3t \end{bmatrix}$

$= \begin{bmatrix} 4\cos 2t - 2\sin 2t - 4\cos 3t + \frac{20}{7}\cos 4t + \sin 4t - \frac{20}{7}\cos 3t \\ \cos 2t - \frac{1}{2}\sin 2t - \cos 3t - \frac{10}{7}\cos 4t - \frac{1}{2}\sin 4t + \frac{10}{7}\cos 3t \end{bmatrix}$

$= \begin{bmatrix} (4\cos 2t - 2\sin 2t) + (\frac{20}{7}\cos 4t + \sin 4t) - 48/7\cos 3t \\ (\cos 2t - \frac{1}{2}\sin 2t) - (\frac{10}{7}\cos 4t + \frac{1}{2}\sin 4t) - 3/7\sin 3t \end{bmatrix}$

$$\boxed{\begin{aligned} x_1 &= 4\cos 2t - 2\sin 2t + 20/7\cos 4t + \sin 4t - 48/7\cos 3t \\ x_2 &= \cos 2t - 1/2\sin 2t - 10/7\cos 4t - 1/2\sin 4t - 3/7\cos 3t \end{aligned}}$$

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j) $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}4&-2\\1&1\end{bmatrix}\begin{bmatrix}y_{1h}+y_{1p}\\y_{2h}+y_{2p}\end{bmatrix} = (y_{1h}+y_{1p})\begin{bmatrix}4\\1\end{bmatrix} + (y_{2h}+y_{2p})\begin{bmatrix}-2\\1\end{bmatrix} = y_{1h}\begin{bmatrix}4\\1\end{bmatrix} + y_{2h}\begin{bmatrix}-2\\1\end{bmatrix} + \begin{bmatrix}4y_{1p}-2y_{2p}\\y_{1p}+y_{2p}\end{bmatrix}$

$= (\cos 2t - \frac{1}{2}\sin 2t)\begin{bmatrix}4\\1\end{bmatrix} + (-\frac{10}{7}\cos 4t - \frac{1}{2}\sin 4t)\begin{bmatrix}-2\\1\end{bmatrix} + \begin{bmatrix}4(-\cos 3t) - 2(\frac{10}{7}\cos 3t)\\(-\cos 3t) + (\frac{10}{7}\cos 3t)\end{bmatrix}$ $\qquad -4 - \frac{20}{7} = -\frac{48}{7}$, $\quad -1 + \frac{10}{7} = \frac{3}{7}$

$$\boxed{\begin{bmatrix}x_1\\x_2\end{bmatrix} = \underbrace{(\cos 2t - \tfrac{1}{2}\sin 2t)}_{y_{1h}}\begin{bmatrix}4\\1\end{bmatrix} + \underbrace{(-\tfrac{10}{7}\cos 4t - \tfrac{1}{2}\sin 4t)}_{y_{2h}}\begin{bmatrix}-2\\1\end{bmatrix} + \cos 3t\underbrace{\begin{bmatrix}-48/7\\3/7\end{bmatrix}}_{\vec{b}_3}}$$

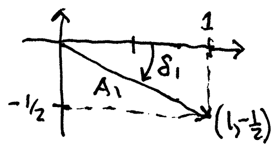
$\omega_1 = 2$ tandem (slow mode)

$\omega_2 = 4$ accordian (fast mode)

$\vec{b}_3 \approx \begin{bmatrix}6.857\\0.429\end{bmatrix}$ need numerical values to plot

accordian

k) $y_{1h} = \cos 2t - \frac{1}{2}\sin 2t$



$A_1 = \sqrt{1 + (-\frac{1}{2})^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} \approx 1.1180$

$\delta_1 = -\arctan(1/2)$ negative acute angle!

≈ -0.4636

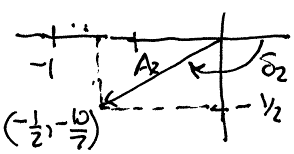
$\approx -26.6°$

$\frac{\delta_1}{2\pi} \approx -0.074$ cycles

(peaks slightly before $t = 0$)

$y_{1h} = \frac{\sqrt{5}}{2}\cos(2t + \arctan\frac{1}{2})$

$y_{2h} = -\frac{10}{7}\cos 4t - \frac{1}{2}\sin 4t$



$A_2 = \sqrt{(-\frac{10}{7})^2 + (-\frac{1}{2})^2}$

$= \frac{\sqrt{449}}{14} \approx 1.5135$

negative obtuse angle! $\quad \delta_2 = -\pi + \arctan\frac{7}{20} \approx -2.8049$

$\approx -160.7°$

$\frac{\delta_2}{2\pi} \approx -0.446$ cycles

(peaks almost half a cycle before $t = 0$)

$y_{2h} = \frac{\sqrt{449}}{14}\cos(4t + \pi - \arctan\frac{7}{20})$

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extra interpretation:

$|A_1\langle 4,1\rangle| = \frac{\sqrt{5}}{2}\sqrt{17} \approx 4.61$ $\qquad |A_2\langle -2,1\rangle| = \frac{\sqrt{449}}{14}\sqrt{5} \approx 3.38$

slow mode has greater displacement from origin in x_1-x_2 plane

$|\vec{b}_3| = \frac{3}{7}\sqrt{257} \approx 6.87$ response mode largest displacement of all

$\left(A_2, \delta_2\right)$ respectively, making a completely labeled diagram in the sinusoidal coefficient plane that supports your work for each case.

l) Optional. Ignore unless you are curious and have adequate time to consider this.
If you plot one cycle of the solution functions you see a maximum departure from the origin in one of the two variables which occurs at $t = \pi$. Identify this variable and its exact and approximate value. Make a plot of the 6 vectors: $\pm A_1 \vec{b}_1, \pm A_2 \vec{b}_2, \pm \vec{b}_3$ along which the 3 independent oscillations take place.

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have not accessed any of the class web pages or any other sites during the exam. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: