MAT2500-01 13S Quiz 5 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

1. a) Use implicit differentiation to evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(x, y, z) = (1, 2, 3)$ of the sphere $x^2 + y^2 + (z-1)^2 = 9$.

b) Write an equation for the tangent plane to this sphere at this point using this information and simplify it to the standard form $ax + by + cz = d$.

2. Evaluate the linear approximation $\mathscr{P}(L, K)$ to the Cobb-Douglas production function $P(L, K)$ $= 1.01\, L^{3/4} K^{1/4}$ at the point $(16, 16)$ and use it to approximate $P(17, 15)$.

3. The surface area of a torus (donut!) of cross-section radius r and circumferential radius R is $S = 2\pi R\,(\pi r^2) = 2\pi^2 r^2 R$. For a torus with $r = 2$ and $R = 3$, use the differential approximation to estimate the percentage increase in surface area if both dimensions increase by 1 percent.

never multiply out unless necessary!!

▶ solution

① a) $\frac{\partial}{\partial x}\left(x^2 + y^2 + (z-1)^2 = 9\right)$

$$2x + 0 + 2(z-1)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{2x}{2(z-1)} = -\frac{x}{z-1}$$

$\frac{\partial}{\partial y}\left(x^2 + y^2 + (z-1)^2 = 9\right)$

$$0 + 2y + 2(z-1)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{2(z-1)} = \frac{-y}{z-1}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2,3)} = \frac{-(1)}{3-1} = \boxed{-\frac{1}{2}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2,3)} = -\frac{2}{3-1} = \boxed{-1}$$

$$z = z_0 + \left.\frac{\partial z}{\partial x}\right|_{x_0 y_0 z_0}(x - x_0) + \left.\frac{\partial z}{\partial y}\right|_{x_0 y_0 z_0}(y - y_0)$$

$$= 3 - \tfrac{1}{2}(x-1) - 1(y-2)$$

$$= 3 - \tfrac{1}{2}x + \tfrac{1}{2} - y + 2 \quad = -\tfrac{1}{2}x - y + \tfrac{11}{2}$$

$$\boxed{\tfrac{1}{2}x + y + z = \tfrac{11}{2} \quad \text{or} \quad x + 2y + 2z = 11}$$

②
$$P(L,K) = 1.01\, L^{3/4} K^{1/4} \qquad P(16,16) = 1.01\; 16^{3/4}\, 16^{1/4} = 1.01 \cdot 8 \cdot 2 = 1.01 \cdot 16$$

$$P_L(L,K) = 1.01\, \tfrac{3}{4} L^{-1/4} K^{1/4} \qquad P_L(16,16) = 1.01\, \tfrac{3}{4}\left(\tfrac{16}{16}\right)^{1/4} = 1.01 \cdot 3/4$$

$$P_K(L,K) = 1.01\, \tfrac{1}{4} L^{3/4} K^{-3/4} \qquad P_K(16,16) = 1.01\, \tfrac{1}{4}\left(\tfrac{16}{16}\right)^{3/4} = 1.01 \cdot 1/4$$

continued

never multiply out unless necessary!
leave 1.01 factor outside.

② (continued)

$$\boxed{\begin{aligned}\mathscr{P}(L,K) &= P(16,16) + P_L(16,16)(L-16) + P_K(16,16)(K-16)\\ &= 1.01\left[16 + \tfrac{3}{4}(L-16) + \tfrac{1}{4}(K-16)\right]\end{aligned}} \quad \boxed{= 1.01\left(\tfrac{3}{4}L + \tfrac{1}{4}K\right)}$$

$$\mathscr{P}(17,15) = 1.01\left[16 + \tfrac{3}{4}(17-16) + \tfrac{1}{4}(15-16)\right]$$

$$= 1.01\left[16 + 3/4 - \tfrac{1}{4}\right] = 1.01\,[16.5]$$

$$\approx \boxed{16.67} \quad (\text{compare with } P(17,16) \approx 16.64)$$

③
$$S = 2\pi^2 r^2 R \qquad \frac{\partial S}{\partial r} = 2\pi^2(2r)R$$

$$dS = 2\pi^2(2r\,dr)R + 2\pi^2 r^2\,dR \qquad \frac{\partial S}{\partial R} = 2\pi^2 r^2$$

$$= 2\pi^2 r\,(2R\,dr + r\,dR)$$

(d obeys product rule!)

$$\frac{dS}{S} = \frac{2\pi^2 r}{2\pi^2 r^2 R}(2R\,dr + r\,dR)$$

$$= \frac{2\,dr}{r} + \frac{dR}{R}$$

$$\left(\frac{dr}{r}\right) = .01, \quad \frac{dR}{R} = .01 \qquad (\text{both increase by } 1\%)$$

$$\frac{2S}{S} = 2(.01) + .01 = .03 \to 3\%$$

$$\boxed{\text{The surface area increases by } 3\% \text{ if its dimensions increase by } 1\%}$$