MAT2500-01 13S Test3 Answers

1 a) $\int_0^{3/2}\int_{\sqrt{3}y}^{\sqrt{9-y^2}} x\,y\,dx\,dy$

$= \int_0^{3/2} \frac{x^2y}{2}\Big|_{x=\sqrt{3}y}^{x=\sqrt{9-y^2}} dy$

$= \int_0^{3/2} \frac{y}{2}((9-y^2)-3y^2)\,dy$

$= \int_0^{3/2} \frac{1}{2}(9y-y^3-3y^3)\,dy$

$= \int_0^{3/2} \frac{1}{2}(9y-4y^3)\,dy$

$= \frac{1}{2}\left(\frac{9}{2}y^2 - \cancel{4}\frac{y^4}{\cancel{4}}\right)\Big|_0^{3/2}$

$= \frac{1}{2}\left(\frac{9}{2}\left(\frac{9}{4}\right) - \frac{81}{16}\right) = \frac{81}{16}(2-1)$

$= \boxed{\frac{81}{32}}$

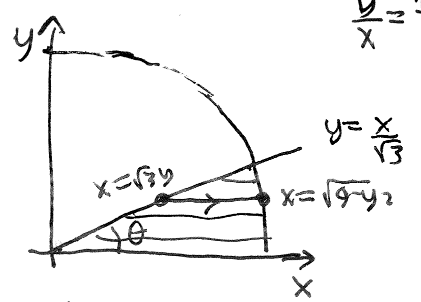
b) $\int_{y=0}^{y=3/2}\int_{x=\sqrt{3}y}^{x=\sqrt{9-y^2}} xy\,dx\,dy$

$x^2 = 9-y^2$

$x^2+y^2=9$

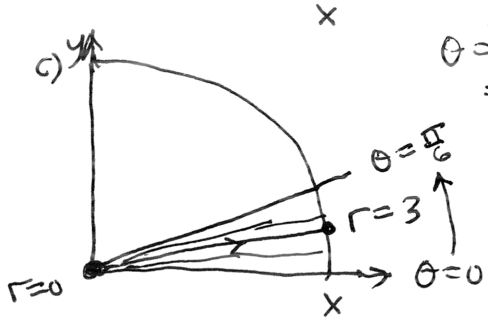
$y = \frac{x}{\sqrt{3}}$ $\quad r=3$

$\frac{y}{x} = \tan\theta = \frac{1}{\sqrt{3}}$



$\theta = \arctan\frac{1}{\sqrt{3}}$
$= \pi/6$

c)



$\int_0^{\frac{\pi}{6}}\int_0^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$

$= \underbrace{\int_0^3 r^3\,dr}\ \underbrace{\int_0^{\frac{\pi}{6}} \underbrace{\sin\theta}_{u}\ \underbrace{\cos\theta\,d\theta}_{du}}$

$\frac{r^4}{4}\Big|_0^3 = \frac{81}{4}$ $\qquad \frac{u^2}{2} \to \frac{\sin^2\theta}{2}\Big|_0^{\pi/6} = \frac{(\sin\pi/6)^2}{2} = \frac{\left(\frac{1}{2}\right)^2}{2}$

$= \frac{1}{8}$

$= \left(\frac{81}{4}\right)\left(\frac{1}{8}\right) = \frac{81}{32}$

2 a) $\int_{x=0}^{x=1}\int_{y=2-2\sqrt{1-x}}^{y=2\sqrt{x}} y\,dy\,dx =$ maple $\boxed{\frac{2}{3}}$

b) $y = 2\sqrt{x} \to x = \frac{y^2}{4}$

c) $y = 2-2\sqrt{1-x} \to \left(\frac{y-2}{2}\right)^2 = 1-x$ $\quad x = 1-\left(\frac{y-2}{2}\right)^2$ $\quad$ " $y-\frac{1}{4}y^2$

intersection: $2\sqrt{x} = 2-2\sqrt{1-x} \xrightarrow{\text{Maple}} x=0,1$

or square: $4x = 4-8\sqrt{1-x}+4(1-x) = 8-4x-8\sqrt{1-x}$

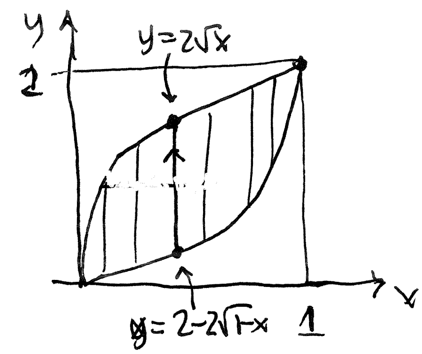
$8x-8 = -8\sqrt{1-x} \to x-1 = -\sqrt{1-x}$

square! $(x-1)^2 = (1-x)$

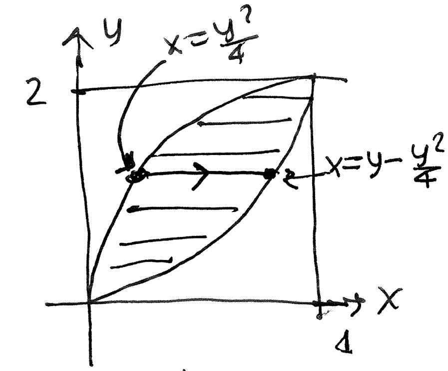
$(x-1)^2+(x-1)=0$

$(x-1)((x-1)+1)=0$

$x(x-1)=0$



$x=0..1$



$y=0..2$

d) $\int_0^2\int_{\frac{y^2}{4}}^{y-\frac{y^2}{4}} y\,dx\,dy$

e) $= \int_0^2 xy\Big|_{x=\frac{y^2}{4}}^{x=y-\frac{y^2}{4}} dy = \int_0^2 y\left(y-\frac{y^2}{4}-\frac{y^2}{4}\right)dy$

$= \int_0^2 \left(y^2-\frac{y^3}{2}\right)dy = \frac{y^3}{3}-\frac{y^4}{8}\Big|_0^2 = \frac{8}{3}-\cancel{\frac{16}{8}}\,2$

$= \frac{8}{3}-\frac{6}{3} = \boxed{\frac{2}{3}}$ ✓ f) yes!

MAT2500-01 13S Test3 Answers (2)

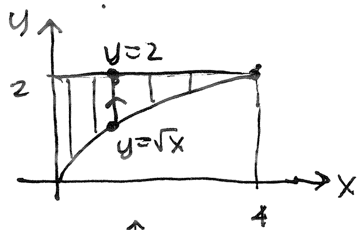
By hand:

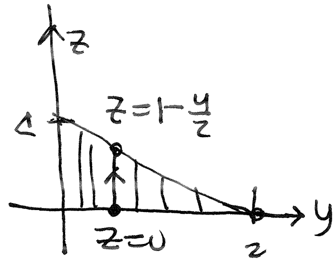
③ a) $\int_{x=0}^{x=4}\int_{y=\sqrt{x}}^{y=2}\int_{z=0}^{z=1-\frac{y}{2}} f(x,y,z)\,dz\,dy\,dx \xrightarrow{f=1} \boxed{\frac{2}{3}} = \int_0^4\int_{\sqrt{x}}^2 \underbrace{z\Big|_{z=0}^{z=1-\frac{y}{2}}}_{1-\frac{y}{2}}\,dy\,dx$

$\underbrace{\int_{\sqrt{x}}^2 \left(1-\frac{y}{2}\right)dy}_{}$: $y-\frac{y^2}{4}\Big|_{y=\sqrt{x}}^{y=2} = \underbrace{(2-1)}_{1}-\left(\sqrt{x}-\frac{x}{4}\right)$

$= x-\left(\frac{x^{3/2}}{3/2}-\frac{x^2}{8}\right)\Big|_{x=0}^{x=4} = 4-\frac{2}{3}(8)+\frac{16}{8}$

$= 6-\frac{16}{3} = \frac{18-16}{3} = 2/3$ ✓

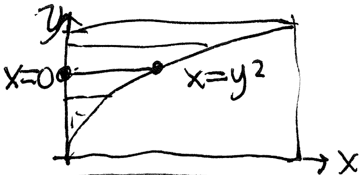




b) $\iiint f\,dx\,dz\,dy$ → same diagram

$y=\sqrt{x}$

x first: floor: $x=0$, ceiling: $x=y^2$



$\boxed{\int_0^2\int_0^{1-\frac{y}{2}}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy} \xrightarrow{f=1} \boxed{\frac{2}{3}} =$

$\int_0^2\int_0^{1-\frac{y}{2}} \underbrace{x\Big|_{x=0}^{x=y^2}}_{y^2}\,dz\,dy$

$y^2z\Big|_{z=0}^{z=1-\frac{y}{2}} = y^2\left(1-\frac{y}{2}\right) = y^2-\frac{y^3}{2}$

$= \left[\frac{y^3}{3}-\frac{y^4}{8}\right]_0^2 = \frac{8}{3}-\frac{16}{8} = \frac{2}{3}$ ✓

c) $\iiint f\,dy\,dz\,dx$

y first: floor, ceiling

$y=\sqrt{x}\,..\,2(1-z)$ in 3-d plot.

~~first diagram~~

while $x=0..4$

intersect $y=\sqrt{x}$, $z=1-\frac{y}{2}$ → $y=2(1-z)$

$\sqrt{x}=2(1-z)$

$x=4(1-z)^2$

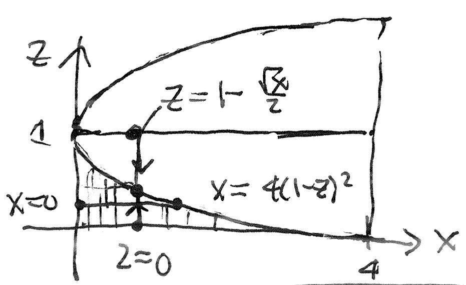
$z=1\to x=0$

$z=0\to x=4$

invert:

$1-z=\frac{\sqrt{x}}{2}$

$z=1-\frac{\sqrt{x}}{2}$



$\boxed{\int_0^4\int_0^{1-\frac{\sqrt{x}}{2}}\int_{\sqrt{x}}^{2(1-z)} f(x,y,z)\,dy\,dz\,dx} \xrightarrow{f=1} \boxed{\frac{2}{3}}$

$= \int_0^4\int_0^{1-\frac{\sqrt{x}}{2}} \underbrace{y\Big|_{y=\sqrt{x}}^{y=2(1-z)}}_{2-2z-x^{1/2}}\,dz\,dx$

$2z-z^2-x^{1/2}z\Big|_{z=0}^{z=1-\frac{x^{1/2}}{2}} = -x^{1/2}\left(1-\frac{x^{1/2}}{2}\right)+2\left(1-\frac{x^{1/2}}{2}\right)-\left(1-\frac{x^{1/2}}{2}\right)^2$

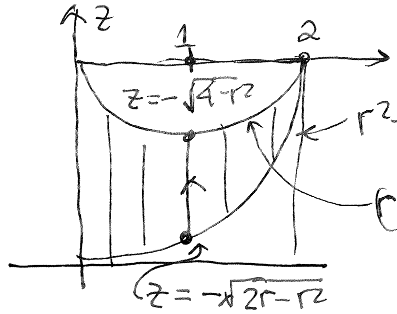
$= \int_0^4 -x^{1/2}+\frac{x}{2}+2-x^{1/2}-1+x^{1/2}-\frac{x}{4}\,dx$

$= x+\frac{x^2}{8}-\frac{2}{3}x^{3/2}\Big|_0^4 = 4+\frac{16}{8}-\frac{2}{3}(8) = 6-\frac{16}{3} = \frac{2}{3}$

oops

MAT2500-01 13S Test 3 Answers (3)

4 a)



$r^2+z^2=4 \to z=-\sqrt{4-r^2}$

$(r-1)^2+z^2=1 \to z=-\sqrt{1-(r-1)^2}$
$=-\sqrt{2r-r^2}$

while $\theta=0..2\pi$, $r=0..2$

$$\boxed{V=\int_0^{2\pi}\int_{-\sqrt{2r-r^2}}^{-\sqrt{4-r^2}} 1\, r\,dz\,d\theta}$$

d) $\to \displaystyle\int_0^{2\pi}\int\int_{-\sqrt{2r-r^2}}^{\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta=-\frac{8\pi}{3}$

$$\bar z=\frac{-8\pi/3}{\pi\left(\frac{16}{3}-\pi\right)}=\boxed{-\frac{8}{16-3\pi}}\approx -1.217$$

b) $\displaystyle=\int_0^{2\pi}\int_0^{2} z\,r\Big|_{z=-\sqrt{2r-r^2}}^{z=-\sqrt{4-r^2}} dr\,d\theta$

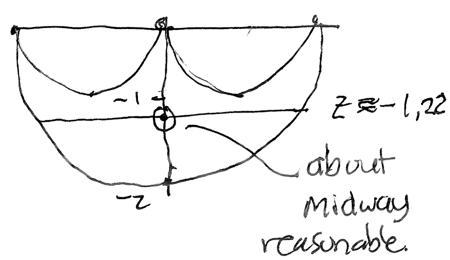
$\swarrow$ 2π

$\underbrace{\int \underbrace{r\sqrt{2r-r^2}}_{\text{cant do without Maple}} - \underbrace{r\sqrt{4-r^2}}_{\text{could do with u-sub}}\,dr}$

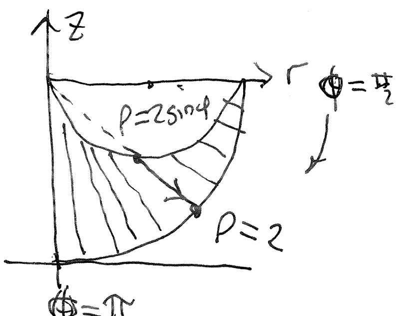
$$=\frac13(2r-r^2)^{3/2}+\frac14(2-2r)\sqrt{2r-r^2}-\frac13(4-r^2)^{3/2}-\frac12\arcsin(r-1)\Big|_0^2$$

$$=\frac13(0)+(0)-0-\frac12\arcsin(1)\quad=\frac83-\underbrace{\arcsin 1}_{\frac{\pi}{2}}$$
$$-0-0+\frac13(8)+\frac12\arcsin(-1)$$

$$=2\pi\left(\frac83-\frac{\pi}{2}\right)=\boxed{\frac{16\pi}{3}-\pi^2}\approx 6.886$$



c)



$(r-1)^2+z^2=1$

$r^2-2r+1+z^2=1$

$\underbrace{(r^2+z^2)}=2r$

$\frac{1}{\rho}\left[\rho^2=2\rho\sin\phi\right]$

$\rho=2\sin\phi$

$\rho=2\sin\phi..2$

$\phi=\frac{\pi}{2}..\pi$

$\theta=0..2\pi$

d) $\to \displaystyle\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_{2\sin\phi}^{2}\rho(\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$=-\frac{8\pi}{3}$

$$\boxed{V=\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_{2\sin\phi}^{2} 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

$\displaystyle=2\pi\int_{\frac{\pi}{2}}^{\pi}\underbrace{\frac{\rho^3}{3}\Big|_{\rho=2\sin\phi}^{\rho=2}\sin\phi}\,d\phi=\frac{16\pi}{3}\int_{\frac{\pi}{2}}^{\pi}\left[\sin\phi-\sin^4\phi\right]d\phi$

$\frac13(-8\sin^3\phi+8)\sin\phi$ $\displaystyle=\frac{16\pi}{3}\left[\left(\frac{\cos\phi}{4}\sin^3\phi+\frac38\cos\phi\right)\sin\phi-\cos\phi-\frac38\phi\right]_{\phi=\pi/2}^{\phi=\pi}$

$$=\frac{16\pi}{3}\left[\frac{+1}{1}-\frac38\pi-\left(-\frac38\left(\frac{\pi}{2}\right)\right)\right]=\frac{16\pi}{3}\left(-\frac{3\pi}{16}\right)=\frac{16\pi}{3}\left(1-\frac{3\pi}{16}\right)=\frac{16\pi}{3}-\pi^2$$

MAT2500-01 13S Test 3 Answers (4)

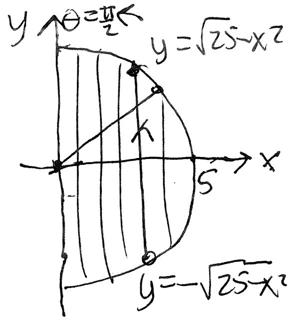
(5) a) $\underbrace{\int_{x=0}^{x=5}\int_{y=-\sqrt{25-x^2}}^{y=\sqrt{25-x^2}}}\int_{z=0}^{z=\sqrt{25-x^2-y^2}} \underbrace{\frac{1}{1+x^2+y^2+z^2}}\,\underbrace{dz\,dy\,dx} \overset{\text{Maple}}{\approx} 11.393$

$\frac{1}{1+\rho^2}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$y^2=25-x^2$

$x^2+y^2=25$

$r=5$



$\theta=-\frac{\pi}{2}$

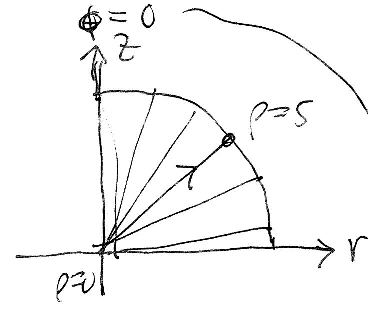
$\theta=-\frac{\pi}{2}\,..\,\frac{\pi}{2}$

$z^2=25-x^2-y^2$

$x^2+y^2+z^2=25$

$\rho=5$

$\phi=\frac{\pi}{2}$



$\phi=\frac{\pi}{2}$

$\rho=0\,..\,5$ while $\phi=0\,..\,\frac{\pi}{2}$

b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^5 \frac{1}{1+\rho^2}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

c) $= \underbrace{\int_0^5 \frac{\rho^2}{1+\rho^2}\,d\rho}\;\underbrace{\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi}\;\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,d\theta} = \boxed{(5-\arctan 5)\pi \approx 11.393}$ ✓

$\rho-\arctan\rho\Big|_0^5 \qquad -\cos\phi\Big|_0^{\pi/2} \qquad \theta\Big|_{-\frac{\pi}{2}}^{\pi/2}=\pi$

$(5-\arctan 5) \qquad = 1$