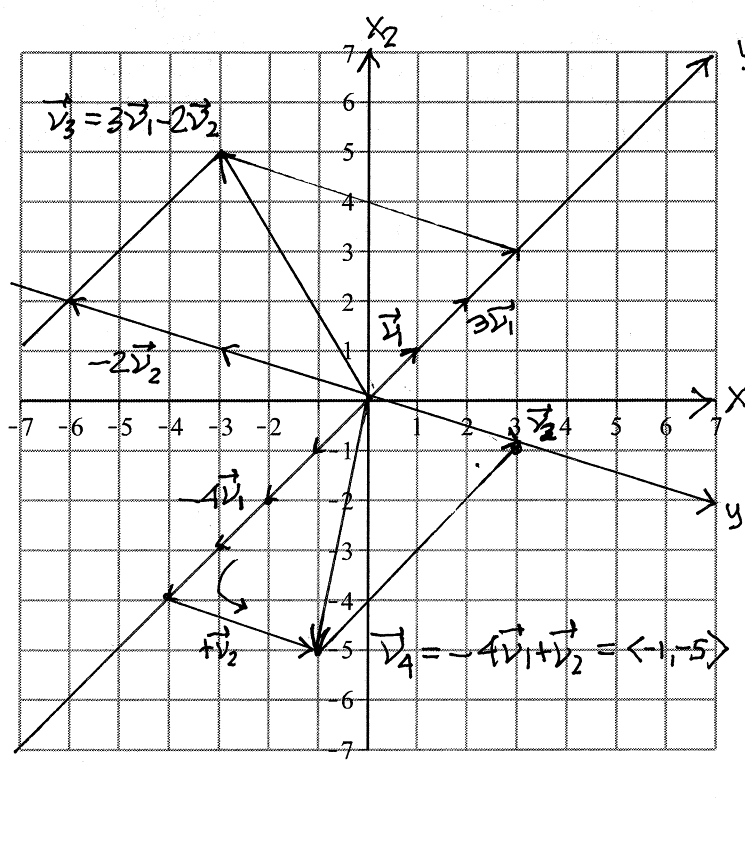
MAT2705-01/02 13F Test 2   Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **You may use technology for row reductions, determinants and matrix inverses.**

1. a) On the grid below, **draw in** arrows representing the vectors $\vec{v}_1 = \langle 1, 1\rangle$ and $\vec{v}_2 = \langle 3, -1\rangle$ and $\vec{v}_3 = \langle -3, 5\rangle$ and **label** them by their symbols. Then **draw in** the parallelogram that graphically expresses $\vec{v}_3$ as a linear combination of $\{\vec{v}_1, \vec{v}_2\}$. **Label** its two sides that intersect at the origin by the corresponding vectors they represent. **Extend** the basis vectors $\{\vec{v}_1, \vec{v}_2\}$ to the corresponding coordinate axes for (y_1, y_2) and **mark** the positive direction with an arrow head and the axis label.
From the grid, read off the coordinates (y_1, y_2) of $\vec{v}_3$ with respect to these two vectors (write them down) and **express** $\vec{v}_3$ as a linear combination of these vectors; **put this equation** at the tip of this vector. **Explain** how you got these numbers.
b) Now write down the matrix equation that enables you to express $\vec{v}_3$ as a linear combination of the other two vectors, solve that system using matrix methods, and then express $\vec{v}_3$ explicitly as a linear combination of those vectors.
c) Check your linear combination by expanding it out to get the original vector. Did you?
d) Does your matrix result agree with part a)?
e) Now using the new coordinate axes, **draw in** the vector $\vec{v}_4$ whose new coordinates are $(y_1, y_2) = (-4, 1)$ and and **label** it by its symbol. Read off its old coordinates (x_1, x_2) from the grid. Do they agree with the linear combination $y_1\vec{v}_1 + y_2\vec{v}_2$?



← The intention was to use the y_1, y_2 axes to plot $\vec{v}_4$: $-4\vec{v}_1$ along y_1 axis from origin, then move $+\vec{v}_2$ along y_2 direction to arrive at tip of $\vec{v}_4$, then read off its (x_1, x_2) coordinates.

I should have requested that you plot the related parallelograms as in this figure. ←

MAT2705-01/02 13F Test 2 Answers

① a) From the diagram: $\vec{v}_3 = 3\vec{v}_1 - 2\vec{v}_2$ has coordinates $(y_1, y_2) = (3, -2)$ wrt the new basis $\{\vec{v}_1, \vec{v}_2\}$.
parallelogram vector addition makes this obvious.

b) $y_1\begin{bmatrix}1\\1\end{bmatrix} + y_2\begin{bmatrix}3\\-1\end{bmatrix} = \boxed{\begin{bmatrix}1&3\\1&-1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}-3\\5\end{bmatrix}}$

soln via inverse matrix:

$$\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}1&3\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}-3\\5\end{bmatrix} = \frac{1}{-1-3}\begin{bmatrix}-1&-3\\-1&1\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}1&3\\1&-1\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-3+15\\-3-5\end{bmatrix} = \frac{1}{4}\begin{bmatrix}12\\-8\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}$$

so $\boxed{\begin{bmatrix}-3\\5\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix} - 2\begin{bmatrix}3\\-1\end{bmatrix}}$

c) $= \begin{bmatrix}3\\3\end{bmatrix} - \begin{bmatrix}6\\-2\end{bmatrix} = \begin{bmatrix}3-6\\3+2\end{bmatrix} = \begin{bmatrix}-3\\5\end{bmatrix}$ ✓

d) They agree! Yay!

e) $\vec{V}_4 = \underbrace{-4\vec{v}_1}_{} + \vec{v}_2$

go along $-y_1$ direction from origin

↑ then add $\vec{v}_2$ to tip ~~to arrive at~~ $\boxed{\langle -1, -5\rangle}$

then draw in $\vec{v}_4$ on diagram.

explicitly:
$\vec{v}_4 = -4\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}-4+3\\-4-1\end{bmatrix} = \boxed{\begin{bmatrix}-1\\-5\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}}$

yes, they agree.

② Note $A = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 | \vec{v}_4 \rangle$

$$= \begin{bmatrix}3&-1&1&0\\2&3&8&1\\-1&4&7&-2\\4&2&8&3\end{bmatrix} \xrightarrow[\text{maple}]{\text{rref}} \begin{array}{cccc} L&L&F&L\\ \end{array} \begin{bmatrix}1&0&1&0\\0&1&2&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

so $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ are linearly independent.

a) $A\vec{x} = \vec{v}_5 = \langle 1, 7, 9, 5\rangle$:

$$\langle A | \vec{v}_5 \rangle = \begin{bmatrix}3&-1&1&0&1\\2&3&8&1&7\\-1&4&7&-2&9\\4&2&8&3&5\end{bmatrix} \xrightarrow[\text{Maple}]{\text{rref}} \begin{array}{c}\begin{bmatrix}1&0&&0&1\\0&1&2&0&2\\0&0&0&1&-1\\0&0&0&0&0\end{bmatrix}\\ \begin{array}{ccccc}x_1&x_2&x_3&x_4&\\L&L&F&L&\end{array}\end{array}$$

particular soln: $\vec{v}_5 = \vec{v}_1 + 2\vec{v}_2 - \vec{v}_4$

gen. soln. $x_3 = t$

$x_1 + x_3 = 1 \qquad x_1 = 1 - t$

$x_2 + 2x_3 = 2 \qquad x_2 = 2 - 2t$

$x_4 = -1$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}1-t\\2-2t\\t\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\0\\-1\end{bmatrix} + t\underbrace{\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix}}_{\vec{x}_h}$$

so $\boxed{\langle 1, 7, 9, 5\rangle = (1-t)\vec{v}_1 + (2-2t)\vec{v}_2 + t\vec{v}_3 - \vec{v}_4}$

2 a) continued. $\boxed{\text{So yes, } \vec{v}_5 \text{ does lie in the span of } \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}.}$

b) $A\vec{x} = \vec{v}_5 = \langle 2, -3, 1, 4\rangle$:

$$\langle A | \vec{v}_5 \rangle = \begin{bmatrix}3&-1&1&0&2\\2&3&8&1&-3\\-1&4&7&-2&1\\4&2&8&3&4\end{bmatrix} \xrightarrow[\text{Maple}]{\text{rref}} \begin{bmatrix}1&0&1&0&0\\0&1&2&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

inconsistent system, no soln ← $0 \neq 1$

$\boxed{\text{so no, } \vec{v}_5 \text{ does not lie in the span of } \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}}$

c) state eqns first: $A\vec{x} = \vec{0}$;

$$\begin{bmatrix}3&-1&1&0\\2&3&8&1\\-1&4&7&-2\\4&2&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\langle A | \vec{0} \rangle = \begin{bmatrix}3&-1&1&0&0\\2&3&8&1&0\\-1&4&7&-2&0\\4&2&8&3&0\end{bmatrix} \xrightarrow[\text{maple}]{\text{rref}} \begin{array}{c}\begin{array}{ccccc}L&L&F&L&\\x_1&x_2&x_3&x_4&\end{array}\\ \begin{bmatrix}1&0&1&0&0\\0&1&2&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}\end{array}$$

$x_3 = t \rightarrow \begin{cases}x_1 + x_3 = 0 \rightarrow x_1 = -t\\ x_2 + 2x_3 = 0 \qquad x_2 = -2t\\ x_4 = 0\end{cases}$

$\boxed{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-t\\-2t\\t\\0\end{bmatrix}}$ just $\vec{x}_h$ from part a) of course

d) $= t\underbrace{\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix}}_{\vec{u} = \langle -1, -2, 1, 0\rangle}$

$\{\vec{u}\}$ is a basis of the soln space
(i.e. one linear relationship among 4 vectors, so 3 are independent.)
$\vec{u}$ = coefficients of linear relationship among the vectors:

e) $\boxed{-\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}}$

f) since at most 3 vectors are linearly independent in this set, their span is 3-dimensional i.e., a $\boxed{\text{hyperplane in } \mathbb{R}^4}$

The leading columns of A provide a basis: $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}}$