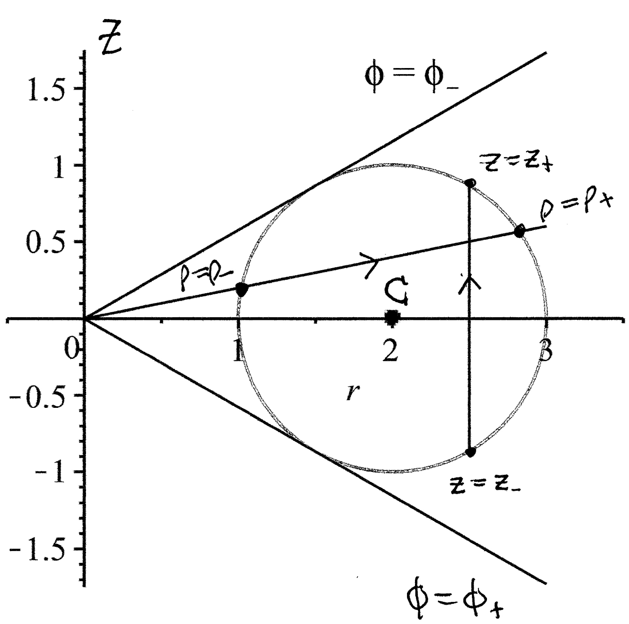
MAT2500-01/04 12S Quiz 9 Pretend Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of each problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).



a) Rotate a circle of radius 1 and center $(2, 0)$ in the r-z half plane about the z-axis to form a torus (donut). Write the equation of this circle.
b) Express z as a function of r for the upper and lower halves of this circle: z_+, z_-. Label the endpoints of the typical vertical cross-section line segment in the diagram by their equations.
c) Set up the triple integral for the volume of the torus in cylindrical coordinates with the integration order $dz\,dr\,d\theta$.
d) Use Maple to evaluate the integral.
e) The theorem of Pappus says that if you take a region of area A in a plane and rotate it about an external axis in the plane, the volume V of revolution it sweeps out equals the product $C\,A$, where C is the circumference of the circle of revolution swept out by its centroid. Does your result agree with this theorem? Explain.

f) Now re-express the equation of the circle in the r-z half plane in spherical coordinates (ρ, ϕ, θ) and simplify it by multiplying out and using the fundamental trig identity. Solve it for ρ as a pair of functions of ϕ using the quadratic formula: ρ_+, ρ_-. Find the value of $\sin(\phi)$ at which these two roots agree (zero discriminant!), denote the corresponding value of ϕ by ϕ_-. Let $\phi_+ = \pi - \phi_-$.

Label the endpoints of the typical radial cross-section in the diagram.
g) Use this information to set up the triple integral in spherical coordinates to evaluate the volume of the torus. Evaluate it using Maple. Did you get the same result?
h) **Optional.** Derive the formula for the volume of the torus if the circle has radius b and center $(a, 0)$. For Maple to evaluate the cylindrical coordinate integral, it requires the extra information:

> *assume*$(b < a)$

solution

a) $\boxed{(r-2)^2+z^2=1}$

or $r^2-4r+4+z^2=1$

$r^2+z^2-4r+3=0$

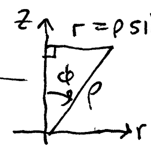
b) $\boxed{z=\pm\sqrt{1-(r-2)^2}\equiv z_\pm}$

c) $\displaystyle\int_0^{2\pi}\int_1^3\int_{-\sqrt{1-(r-2)^2}}^{\sqrt{1-(r-2)^2}} r\,dz\,dr\,d\theta$

d) $= 4\pi^2$ (Maple)

e) $\left.\begin{array}{l} C=2\pi(2)=4\pi \\ A=\pi(1)^2=\pi \end{array}\right\}$ $V=4\pi(\pi)=4\pi^2$ ✓

f) $\underbrace{r^2+z^2}_{\rho^2}-4\underbrace{r}_{\rho\sin\phi}+3=0 \leftarrow r=\rho\sin\phi$



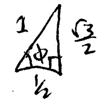
$\boxed{\rho^2-4\rho\sin\phi+3=0}$

$\rho=\boxed{\rho_\pm}=\dfrac{4\sin\phi\pm\sqrt{16\sin^2\phi-4(3)}}{2}=\boxed{2\sin\phi\pm\sqrt{4\sin^2\phi-3}}$

$\rho_+=\rho_-:\ 4\sin^2\phi-3=0$

$\sin\phi=\pm\frac{\sqrt3}{2}$

$\phi=\pm\frac{\pi}{3}\rightarrow \boxed{\frac{\pi}{3}=\phi_-}$ $\boxed{\phi_+=\frac{2\pi}{3}}$



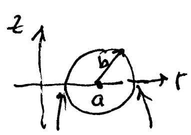
g) $\displaystyle\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_{\rho_-}^{\rho_+}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$= 4\pi^2$ (Maple) ✓

h) $V=\displaystyle\int_0^{2\pi}\int_{a-b}^{a+b}\int_{-\sqrt{b^2-(r-a)^2}}^{\sqrt{b^2-(r-a)^2}} r\,dz\,dr\,d\theta$

$=\boxed{4\pi^2ab^2}$ (Maple)

$=(2\pi a)(\pi b^2)$ ✓



$(r-a)^2+z^2=b^2$

$z=\pm\sqrt{b^2-(r-a)^2}$