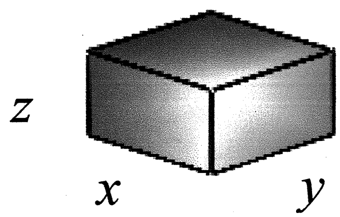
MAT2500-01/04 12S Quiz 7 Print Name (Last, First) ______________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of each problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).



A package in the shape of a rectangular box can be mailed by the USPS if the sum of its length and its girth (the perimeter of a cross-section perpendicular to the length) is at most 108 inches = 9 feet. Find the dimensions of the package with the largest volume that can be mailed. You may work the problem in inches or feet.

Hint: let z be the length, so the girth is 2 (x+y).

a) What is the volume function V, as a function of the dimensions $x > 0, y > 0, z > 0$ suggested below the figure?

b) What is the constraint among these dimensions so that the box has maximum length plus girth allowed by the USPS?

c) Choosing to eliminate z using the constraint, what is the volume function $V(x, y)$ as a function of the remaining two variables?

d) This problem must have an answer, so if you find one critical point, that will be the local and global max. Write down the equations for the critical point, factoring out common factors from each equation, ignoring factors which are positive to solve them. What is the unique critical point you find?

e) What is the value of z and V for this critical point?

f) Check that indeed your function $V(x, y)$ has a local max at your critical point.

g) Answer the word problem with a complete sentence that does not use any mathematical symbols.

▶ solution

a) $\boxed{V = xyz} > 0$

b) $\boxed{z + 2(x+y) = 108}$ inches or $\boxed{z + 2(x+y) = 9}$ feet

c) $z = 108 - 2(x+y)$ $\qquad$ $z = 9 - 2(x+y)$

$\boxed{V(x,y) = xy(108-2(x+y)) = 108xy - 2x^2y - 2xy^2}$ inches $\qquad$ $\boxed{V(x,y) = xy(9-2(x+y)) = 9xy - 2x^2y - 2xy^2}$ feet

d) $\frac{\partial V}{\partial x} = 108y - 4xy - 2y^2 = 2y(54-2x-y) = 0$ $\rightarrow$ $9y - 4xy - 2y^2 = y(9-4x-2y) = 0$

$\frac{\partial V}{\partial y} = 108x - 2x^2 - 4xy = 2x(54-x-2y) = 0$ $\rightarrow$ $9x - 2x^2 - 4xy = x(9-2x-4y) = 0$

$54-2x-y=0$ $\qquad$ $2x+y=54$ $\qquad$ $x = 54-2y = 54-36 = 18$ $\qquad$ $z = 108-2(18+18) = 36$

$54-x-2y=0$ $\qquad$ $2[x+2y=54]$

$\ominus$ $0 + (1-4)y = -54$

$y = 54/3 = 18$

$\boxed{(x,y) = (18,18)}$ $\qquad$ ... $(x,y) = (\frac{3}{2}, \frac{3}{2})$

e) $\boxed{z = 36 \text{ (inches)} \quad V = 18 \cdot 18 \cdot 36 = 11664 (\text{inch}^3)}$ $\qquad$ $\boxed{z = 3 \text{ (ft)} \quad V = \frac{3}{2} \cdot \frac{3}{2} \cdot 3 = \frac{27}{4} = 6\,3/4 = 6.75 \text{ (ft}^3)}$ ← more reasonable volume unit

f) $V_{xx} = 0 - 4y - 0 = -4y \xrightarrow{(18,18)} -4(18) < 0$ ✓

$V_{yy} = 0 - 0 - 4x = -4x \rightarrow -4(18) < 0$ ✓

$V_{xy} = 108 - 4x - 4y \rightarrow 108 - 8(18) = -36$

$V_{xx}V_{yy} - V_{xy}^2 \rightarrow 4^2 18^2 - 36^2 = 3888 > 0$ confirms local max.

g) The package with the largest dimensions that can be mailed with the USPS has length 36 in (= 3 ft) and equal girth dimensions of 18 in (1.5 ft).