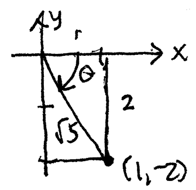
MAT2500-01/04 12S FINAL EXAM ANSWERS

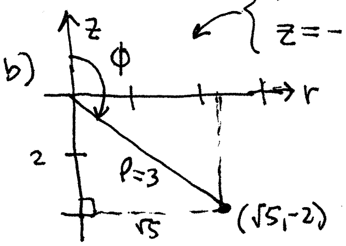
① a) $(x,y,z) = (1,-2,-2)$



$\theta = -\arctan\frac{2}{1}$
$= -\arctan 2$
$\approx -63.4°$

$\begin{cases} r = \sqrt{1^2+(-2)^2} = \sqrt{5} \approx 2.236 \\ z = -2 \end{cases}$

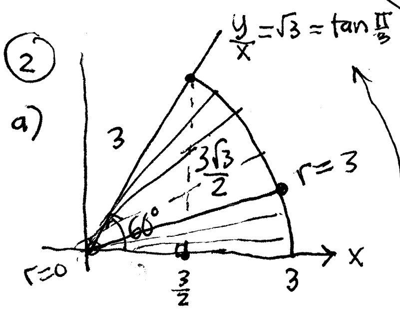
b)



$\rho = \sqrt{5+(-2)^2} = \sqrt{9} = 3$
$= \sqrt{1^2+(-2)^2+(-2)^2} =$

$\phi = \arccos\left(\frac{-2}{3}\right) = -\arccos\left(\frac{2}{3}\right)$
$= \pi - \arctan\left(\frac{\sqrt{5}}{2}\right)$
$\approx 131.8°$

② a)



$\theta = 0..\frac{\pi}{3}$
$r = 0..3$

inequalities:
$\boxed{\begin{matrix} 0 \le \theta \le \pi/3 \\ 0 \le r \le 3 \end{matrix}}$

b) $\langle A, A_x, A_y\rangle = \int_0^{\pi/3}\int_0^3 \langle 1, r\cos\theta, r\sin\theta\rangle\, r\,dr\,d\theta$

$= \left\langle \int_0^{\pi/3}\int_0^3 r\,dr\,d\theta,\ \int_0^{\pi/3}\int_0^3 r^2\cos\theta\,dr\,d\theta,\ \int_0^{\pi/3}\int_0^3 r^2\sin\theta\,\cancel{dr\,d\theta} \right\rangle$

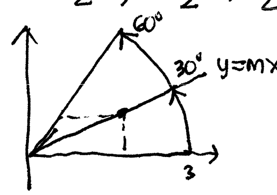
$= \left\langle \int_0^{\pi/3} d\theta \int_0^3 r\,dr,\quad \int_0^{\pi/3}\cos\theta\,d\theta \int_0^3 r^2\,dr,\quad \int_0^{\pi/3}\sin\theta\,d\theta\int_0^3 r^2\,dr \right\rangle$

$\frac{\pi}{3}\left(\frac{r^2}{2}\Big|_0^3\right) \qquad +\sin\theta\Big|_0^{\pi/3} \quad \frac{r^3}{3}\Big|_0^3 \qquad -\cos\theta\Big|_0^{\pi/3} \quad \frac{r^3}{3}\Big|_0^3$

$= \left\langle \frac{\pi}{3}\left(\frac{9}{2}\right),\ \underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}}\cdot 9,\ \left(1-\underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}}\right)9 \right\rangle$

$= \left\langle \frac{3}{2}\pi,\ \frac{9\sqrt{3}}{2},\ \frac{9}{2}\right\rangle$ so $\langle \bar{x}, \bar{y}\rangle = \left\langle \frac{9\sqrt{3}}{2}\Big/\frac{3}{2}\pi,\ \frac{9}{2}\Big/\frac{3}{2}\pi \right\rangle$

$= \left\langle \frac{3\sqrt{3}}{\pi},\ \frac{3}{\pi}\right\rangle \approx \langle 1.65, 0.95\rangle$

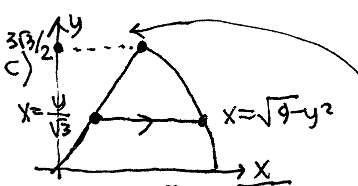


$m = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$ ✓

expect it to lie along bisector, beyond midpoint:

$r \approx \frac{3}{\pi}\sqrt{3+1} = \frac{6}{\pi} \approx 1.91 > \frac{3}{2} = 1.5$

c)



$x^2+y^2=9 \wedge y=\sqrt{3}x \rightarrow$ (intersection)
$x^2+3x^2=9 \rightarrow x^2 = 9/4,\ x=3/2$
$y = 3\sqrt{3}/2$

$A_x = \int_0^{3\sqrt{3}/2}\int_{y/\sqrt{3}}^{\sqrt{9-y^2}} x\,dx\,dy = \int_0^{3\sqrt{3}/2} \frac{x^2}{2}\Big|_{x=y/\sqrt{3}}^{x=\sqrt{9-y^2}} dy$

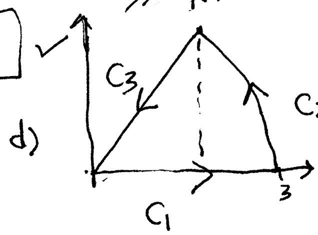
② b) continued

$= \frac{1}{2}\int_0^{3\sqrt{3}/2} (9-y^2) - \frac{y^2}{3}\,dy = \underbrace{\frac{1}{2}\left(9y - \frac{4}{3}\left(\frac{y^3}{3}\right)\right)}_{\frac{1}{2}y\left(9-\frac{4}{9}y^2\right)}\Big|_0^{3\sqrt{3}/2}$

$= \frac{1}{2}\left(\frac{3\sqrt{3}}{2}\right)\left(9 - \frac{4}{9}\cdot\frac{27}{4}\right)$

$= \boxed{\frac{9\sqrt{3}}{2}}$ ✓

d)



$C_1: \vec{r} = \langle x, y\rangle = \langle t, 0\rangle,\ t = 0..3$
$\vec{r}' = \langle 1, 0\rangle, \quad |\vec{r}'(t)| = 1$

$C_2: \vec{r} = \langle 3\cos t, 3\sin t\rangle,\ t = 0..\pi/3$
$\vec{r}' = \langle -3\sin t, 3\cos t\rangle,\ |\vec{r}'| = 3$

$C_3: \vec{r} = \langle t, \sqrt{3}t\rangle,\ t = 0..3/2$ but reverse direction
$\vec{r}' = \langle 1, \sqrt{3}\rangle,\ |\vec{r}'| = 2$

$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_0^3 \langle t, t(0)\rangle\cdot\langle 1, 0\rangle\,dt$
$= \int_0^3 t\,dt = t^2/2\Big|_0^3 = \frac{9}{2}$

$-\int_{C_3}\vec{F}\cdot d\vec{r} = -\int_0^{3/2}\langle t, t(\sqrt{3}t)\rangle\cdot\langle 1, \sqrt{3}\rangle\,dt$
$= -\int_0^{3/2} t + 3t^2\,dt = -\left(\frac{t^2}{2}+t^3\right)\Big|_0^{3/2}$
$= -\left(\frac{1}{2}\cdot\frac{9}{4}+\frac{9}{4}\cdot\frac{3}{2}\right) \qquad = -\frac{9}{2}$

$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_0^{\pi/3}\langle 3\cos t, (3\cos t)(3\sin t)\rangle \cdot\langle -3\sin t, 3\cos t\rangle\,dt$

$= \int_0^{\pi/3} -9\cos t\sin t + 27\cos^2 t\sin t\,dt$
$\int +9u\,du - 27u^2\,du = \frac{9}{2}u^2 - 9u^3$

$= \frac{9}{2}\cos^2 t - 9\cos^3 t\Big|_0^{\pi/3} = \frac{9}{2}\left(\frac{1}{4}\right) - 9\left(\frac{1}{8}\right) - \left(\frac{9}{2}-9\right) = \frac{9}{2}$

$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} - \int_{C_3}\vec{F}\cdot d\vec{r}$

$= \frac{9}{2}+\frac{9}{2}-\frac{9}{2} = \boxed{\frac{9}{2}}$

e) $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x) = y$

$\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA = \iint_R y\,dA = A_y = \boxed{\frac{9}{2}}$

from above

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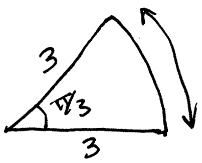
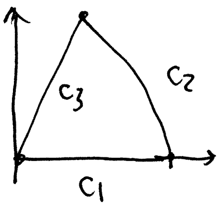
② f)



$3\left(\frac{\pi}{3}\right)$

$\boxed{S = 3+3+3\left(\frac{\pi}{3}\right) = 6+\pi}$

g)



same parametrizations as before.

$ds = |\vec{r}\,'|dt$

$C_1 : y(t) = 0, \quad \int_{C_1} y\, ds = 0$

$C_2 : y(t) = 3\sin t, \quad ds = 3dt : \int_0^{\pi/3} 3\sin t\,(3dt)$

$= -9\cos t\Big|_0^{\pi/3} = 9\left(-\frac{1}{2}+1\right) = \frac{9}{2}$

$C_3 : y(t) = \sqrt{3}\,t, \quad ds = 2dt : \int_0^{3/2} (\sqrt{3}t)(2dt)$

$= 2\sqrt{3}\,\frac{t^2}{2}\Big|_0^{3/2} = \sqrt{3}\left(\frac{9}{4}\right) = \frac{9\sqrt{3}}{4}$

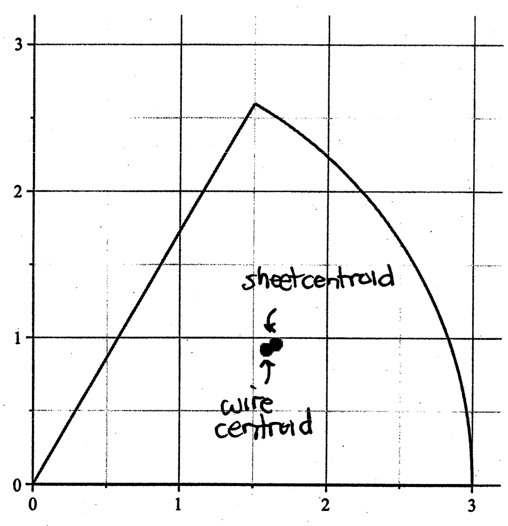
$\int_C y\,ds = \int_{C_1} y\,ds + \int_{C_2} y\,ds + \int_{C_3} y\,ds = \frac{9}{2} + \frac{9\sqrt{3}}{4}$

$\bar{y} = \frac{\int_C y\,ds}{S} = \boxed{\frac{\frac{9}{2}+\frac{9\sqrt{3}}{4}}{6+\pi} \approx 0.919}$

$\bar{x} = \sqrt{3}\,\bar{y} = \boxed{\frac{\frac{9}{2}\sqrt{3}+\frac{27}{4}}{6+\pi} \approx 1.591}$

Compare $\langle 1.59, 0.92 \rangle$ with: $\langle 1.65, 0.95 \rangle$

so the wire centroid is just a hair closer to the origin along the angle bisector compared to the sheet centroid



③ b) $\vec{F} = \langle (1+xy)e^{xy}, e^y + x^2 e^{xy} \rangle$

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\left(e^y + x^2 e^{xy}\right) - \frac{\partial}{\partial y}\left((1+xy)e^{xy}\right)$

$= 0 + 2xe^{xy} + x^2 e^{xy}y - (0+x)e^{xy} - (1+xy)e^{xy}x$

$= (2x + x^2y - x - x - x^2y)e^{xy} = 0\,e^{xy} = 0$ ✓

c) $\vec{F} = \vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$:

$\frac{\partial f}{\partial x} = (1+xy)e^{xy} \longrightarrow f = \int (1+xy)e^{xy}\,dx \overset{\text{Maple}}{=} xe^{xy} + \underbrace{C(y)}_{\text{added by hand}}$

$\frac{\partial f}{\partial y} = e^y + x^2 e^{xy}$

$\frac{\partial f}{\partial y} = xe^{xy}x + C'(y)$

$\cancel{x^2 e^{xy}} + C'(y) = e^y + \cancel{x^2 e^{xy}}$

$C'(y) = e^y \longrightarrow C(y) = \int e^y\,dy = e^y + k$

$\therefore \boxed{f = xe^{xy} + e^y + k}$

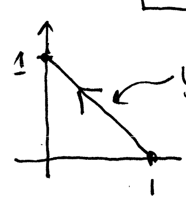
we can set the constant $k=0$ since only differences in potential matter.

d) $\int_C \vec{F}\cdot d\vec{r} = f(0,1) - f(1,0)$

$= 0e^{0(1)} + e^1 - (1e^{1\cdot 0} + e^0)$

$= \boxed{e-2}$

d)



$\vec{r} = \langle x,y \rangle = \langle t, 1-t \rangle, \quad t = 0..1$

but this goes in wrong direction

$\vec{r}\,' = \langle 1,-1 \rangle$

$\vec{F}(\vec{r}) = \langle (1+t(1-t))e^{t(1-t)}, e^{1-t} + t^2 e^{t(1-t)} \rangle$

$\vec{F}(\vec{r})\cdot\vec{r}\,' = 1 + t(1-t)e^{t(1-t)} - e^{1-t} - t^2 e^{t(1-t)}$

$\int_C \vec{F}\cdot d\vec{r} = -\int_0^1 \vec{F}(\vec{r})\cdot\vec{r}\,'\,dt$

$= \int_0^1 -1 - t(1-t)e^{t(1-t)} + e^{1-t} + t^2 e^{t(1-t)}\,dt$

$\overset{\text{Maple}}{=} \boxed{e-2}$ ✓

or $\vec{r} = \vec{r}_1 + t(\vec{r}_2 - \vec{r}_1) \quad t = 0..1$

$= \langle 1,0 \rangle + t(\langle 0,1 \rangle - \langle 1,0 \rangle) = \langle 1-t, t \rangle$

rest very similar...