MAT2705-01/02 12F Final Exam Answers (1)

① a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \begin{bmatrix} -10 & 2 \\ 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -42\cos 2t \\ 21\cos 2t \end{bmatrix}$

$= \begin{bmatrix} -10x_1 + 2x_2 - 42\cos 2t \\ 3x_1 - 15x_2 + 21\cos 2t \end{bmatrix}$

$$\boxed{\begin{aligned} x_1'' &= -10x_1 + 2x_2 - 42\cos 2t \\ x_2'' &= 3x_1 - 15x_2 + 21\cos 2t \\ x_1(0) &= 0 = x_2(0),\ x_1'(0) = 0 = x_2'(0) \end{aligned}}$$

maple

$$\boxed{\begin{aligned} x_1 &= -7\cos 2t + 6\cos 3t + \cos 4t \\ x_2 &= 3\cos 3t - 3\cos 4t \end{aligned}}$$

b) $A = \begin{bmatrix} -10 & 2 \\ 3 & -15 \end{bmatrix}$

$0 = |A - \lambda I| = \begin{vmatrix} -10-\lambda & 2 \\ 3 & -15-\lambda \end{vmatrix} = (\lambda+10)(\lambda+15) - 6$

$= \lambda^2 + 25\lambda + 150 - 6 = \lambda^2 + 25\lambda + 144$

$\to \lambda = -9, -16$ maple (quad formula!)

$\lambda = -9$: $A + 9I = \begin{bmatrix} -10+9 & 2 \\ 3 & -15+9 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

$\xrightarrow{\text{rref}} \begin{array}{cc} L & F \end{array}$ $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{array}{l} x_1 = 2t \\ x_2 = t \end{array}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda = -16$: $A + 16I = \begin{bmatrix} -10+16 & 2 \\ 3 & -15+16 \end{bmatrix} = \begin{bmatrix} +6 & 2 \\ 3 & 1 \end{bmatrix}$

$\xrightarrow{\text{rref}} \begin{array}{cc} L & F \end{array}$ $\begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{array}{l} x_1 = -1/3\, t \\ x_2 = t \end{array}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \xrightarrow{t=3} \vec{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\boxed{\begin{aligned} \lambda &= -9, -16 \\ B &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \end{aligned}}$ $\boxed{B^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}}$ $\boxed{\begin{aligned} A_D &= B^{-1}AB \\ &= \begin{bmatrix} -9 & 0 \\ 0 & -16 \end{bmatrix} \end{aligned}}$

c) $\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $m_1 = \boxed{\frac{1}{2}}$

$\vec{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $m_2 = \frac{3}{-1} = \boxed{-3}$

d) $\vec{x} = B\vec{y}$, $\vec{y} = B^{-1}\vec{x}$

$B^{-1}[\vec{x}'' = A\vec{x} + \vec{F}]$

$\vec{y}'' = A_D\vec{y} + B^{-1}\vec{F}$

$B^{-1}F = \frac{\cos 2t}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -42 \\ 21 \end{bmatrix}$

$= \cos 2t \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -6 \\ 3 \end{bmatrix}$

$= \cos 2t \begin{bmatrix} -15 \\ 12 \end{bmatrix}$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -9 & 0 \\ 0 & -16 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \cos 2t \begin{bmatrix} -15 \\ 12 \end{bmatrix} = \begin{bmatrix} -9y_1 - 15\cos 2t \\ -16y_2 + 12\cos 2t \end{bmatrix}$

$y_1'' = -9y_1 - 15\cos 2t$ $\quad y_1'' + 9y_1 = -15\cos 2t$

$y_2'' = -16y_2 + 12\cos 2t$ $\quad y_2'' + 16y_2 = 12\cos 2t$

① d) continued

$y_{1h} = c_1\cos 3t + c_2\sin 3t \qquad y_{1p} = c_5\cos 2t + c_6\sin 2t$

$y_{2h} = c_3\cos 4t + c_4\sin 4t \qquad y_{2p} = c_7\cos 2t + c_8\sin 2t$

9 $[y_{1p} = c_5\cos 2t + c_6\sin 2t]$

0 $[y_{1p}' = -2c_5\sin 2t + 2c_6\cos 2t]$

1 $[y_{1p}'' = -4c_5\cos 2t - 4c_6\sin 2t]$

$y_{1p}'' + 9y_{1p} = \underbrace{(9-4)}_{5} c_5\cos 2t + \underbrace{(9-5)}_{5} c_6\sin 2t = -15\cos 2t$

$5c_5 = -15 \qquad 5c_6 = 0 \qquad c_5 = -3,\ c_6 = 0$

16 $[y_{2p} = c_7\cos 2t + c_8\sin 2t]$

0 $[y_{2p}' = -2c_7\sin 2t + 2c_8\cos 2t]$

1 $[y_{2p}'' = -4c_7\cos 2t - 4c_8\sin 2t]$

$y_{2p}'' + 16y_{2p} = \underbrace{(16-4)}_{12} c_7\cos 2t + \underbrace{(16-4)}_{12} c_8\sin 2t = 12\cos 2t$

$12c_7 = 12 \qquad 12c_8 = 0 \qquad \begin{array}{l} c_7 = 1 \\ c_8 = 0 \end{array}$

$y_{1p} = -3\cos 2t, \quad y_{2p} = \cos 2t$

$$\boxed{\begin{aligned} y_1 &= c_1\cos 3t + c_2\sin 3t - 3\cos 2t \\ y_2 &= c_3\cos 4t + c_4\sin 4t + \cos 2t \end{aligned}}$$ gen soln uncoupled variables

e) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1 C_3 + c_2 S_3 - 3\cos 2t \\ c_3 C_4 + c_4 S_4 + \cos 2t \end{bmatrix}$

$= \begin{bmatrix} 2(c_1 C_3 + c_2 S_3 - 3\cos 2t) - (c_3 C_4 + c_4 S_4 + \cos 2t) \\ (c_1 C_3 + c_2 S_3 - 3\cos 2t) + 3(c_3 C_4 + c_4 S_4 + \cos 2t) \end{bmatrix}$

$= \begin{bmatrix} 2c_1\cos 3t + 2c_2\sin 3t - c_3\cos 4t - c_4\sin 4t - 7\cos 2t \\ c_1\cos 3t + c_2\sin 3t + 3c_3\cos 4t + 3c_4\sin 4t \end{bmatrix}$

$$\boxed{\begin{aligned} x_1 &= 2c_1\cos 3t + 2c_2\sin 3t - c_3\cos 4t - c_4\sin 4t - 7\cos 2t \\ x_2 &= c_1\cos 3t + c_2\sin 3t + 3c_3\cos 4t + 3c_4\sin 4t \end{aligned}}$$

gen soln. BUT NOT YET – IVP conditions!

Maple's gen soln corresponds to eigenvectors with first component 1 instead of second:

$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1/2 & -3 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1\cos 3t + c_2\sin 3t - 3\cos 2t \\ c_3\cos 4t + c_4\sin 4t + \cos 2t \end{bmatrix}$

$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3c_1\sin 3t + 3c_2\cos 3t + 6\sin 2t \\ -4c_3\sin 4t + 4c_4\cos 4t - 2\sin 2t \end{bmatrix}$

$\vec{x}(0) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1 - 3 \\ c_3 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{array}{l} c_1 = 3 \\ c_3 = -1 \end{array}$

$\vec{x}'(0) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3c_2 \\ 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{array}{l} c_2 = 0 \\ c_4 = 0 \end{array}$

$$\boxed{\begin{aligned} x_1 &= 6\cos 3t + \cos 4t - 7\cos 2t \\ x_2 &= 3\cos 3t - 3\cos 4t \end{aligned}}$$ $\left(\begin{aligned} y_1 &= 3\cos 3t - 3\cos 2t \\ y_2 &= -\cos 4t + \cos 2t \end{aligned}\right)$

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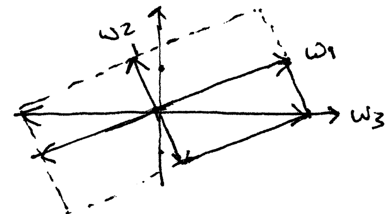
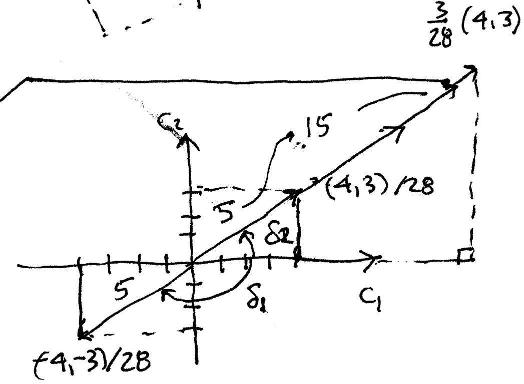
① f) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cos 3t \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \cos 4t \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \cos 2t \begin{bmatrix} -7 \\ 6 \end{bmatrix}$ → g) $\pm \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \pm \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \pm \begin{bmatrix} -7 \\ 0 \end{bmatrix}$

$$= \underbrace{\underset{\text{tandem}}{3\cos 3t\, \vec{b}_1} - \underset{\text{accordian}}{\cos 4t\, \vec{b}_2}}_{\vec{X}_h} + \underbrace{\underset{\text{response}}{\cos 2t\, \vec{b}_3}}_{\vec{X}_p}$$

modes:



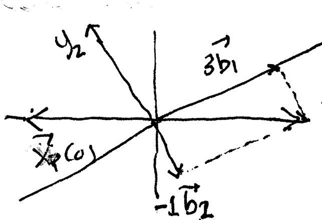
$\boxed{\begin{matrix}\omega_1 = 3 \\ \text{slow} \\ \text{tandem}\end{matrix}}$ $\boxed{\begin{matrix}\omega_2 = 4 \\ \text{fast} \\ \text{accordian}\end{matrix}}$ $\omega_3 = 2$ driving frequency



tickmarks 1/28

g) $\vec{x}_h(0) = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \underset{y_1(0)}{3}\vec{b}_1 - \underset{y_2(0)}{1}\vec{b}_2 = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$

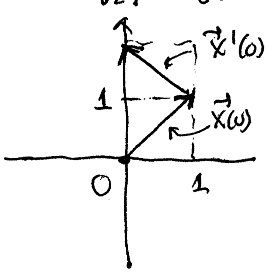
$\vec{x}_p(0) = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$



$\vec{x}_h(0) = 3\vec{b}_1 - 1\vec{b}_2 = -\vec{x}_p(0)$

$\boxed{(y_1, y_2) = (3, -1)}$

h) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \begin{bmatrix} -10 & 2 \\ 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



$A_2 = \frac{3}{28}\sqrt{4^2+3^2} = \frac{3}{28} \cdot 5 = \boxed{\frac{15}{28}}$

$\delta_2 = \boxed{\arctan 3/4}$ (first quad)

$A_1 = \frac{1}{28}\sqrt{4^2+3^2} = \boxed{\frac{5}{28}}$

$\delta_1 = \boxed{-\pi + \arctan 3/4}$ (third quad)

$\frac{A_2}{A_1} = \frac{15/28}{5/28} = \boxed{3}$

$\delta_2 - \delta_1 = \arctan 3/4 - (-\pi + \arctan 3/4)$
$= \boxed{\pi}$

opposite directions since eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ has opp signed components in ratio 1:3, no surprise!!

$T_{\text{accordian}} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$

$x_1 = \frac{8}{7}\cos 3t - \frac{4}{21}\sin 3t - \frac{1}{7}\cos 4t - \frac{3}{28}\sin 4t$

$x_2 = \frac{4}{7}\cos 3t - \frac{2}{21}\sin 3t + \frac{3}{7}\cos 4t + \frac{9}{28}\sin 4t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{4}{7}\left(\cos 3t - \frac{1}{6}\sin 3t\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{7}\left(\cos 4t + \frac{3}{4}\sin 4t\right)\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

faster mode:

$$= \underbrace{\frac{2}{21}(6\cos 3t - \sin 3t)\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\text{ignore for rest of problem}} + \frac{1}{28}(4\cos 4t + 3\sin 4t)\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

x_1: $\frac{1}{28}(-4, -3)$

x_2: $\frac{3}{28}(4, 3)$

fast mode contributions:

$\boxed{\begin{aligned} x_2 &\sim \tfrac{1}{28}(-4\cos 4t - 3\sin 4t) = \tfrac{5}{28}\cos(4t - \arctan 3/4) \\ x_1 &\sim \tfrac{3}{28}(4\cos 4t + 3\sin 4t) = \tfrac{15}{28}\cos(4t - \arctan 3/4 + \pi) \end{aligned}}$