MAT2705-03/04 11S Quiz 5 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. $-3x_1 - 6x_2 + 4x_3 + 7x_4 = -10$
   $2x_1 + 4x_2 - 2x_3 - 2x_4 = 6$

a) Write down the coefficient matrix **A**, the RHS matrix $\vec{\mathbf{b}}$ and the augmented matrix $\mathbf{C} = \langle \mathbf{A} \mid \vec{\mathbf{b}} \rangle$ for this linear system of equations.

b) With technology (identify your choice!), reduce this matrix **C** step by step to its ReducedRowEchelonForm avoiding fractions (4 easy steps!), recording the intermediate matrices and row operations for each step (as in $R_1 \leftrightarrow R_2$, $R_3 \rightarrow R_3 + 2R_1$, $R_1 \rightarrow \frac{1}{2} R_1$).

c) Write out the equations that correspond to the reduced matrix. Identify the leading variables and the free variables and solve. State your solution in the scalar form: $x_1 = \ldots, x_2 = \ldots$, etc .

d) Now examine the solution for the corresponding homogeneous problem with zero right hand sides:

$-3x_1 - 6x_2 + 4x_3 + 7x_4 = 0$

$2x_1 + 4x_2 - 2x_3 - 2x_4 = 0$

Its solution is the part of the previous solution which has only the parameter terms, setting the constant terms which come from the right hand side to zero.

State this solution in column matrix ("vector") form $\vec{x} = \ldots$ and then re-express it as an arbitrary linear combination of fixed vectors by factoring out the vector of coefficients of each of the free parameters in the solution.

2. $2x_1 + x_2 = -2$, $4x_1 + 3x_2 = 4$ a) Write this linear system in the matrix form $\mathbf{A}\vec{x} = \vec{b}$.

b) Write down the inverse coefficient matrix using technology or your memory if good enough, but then verify that its product with **A** is the identity matrix. Show the matrix multiplication steps by hand (sums of products before simplifying) to prove that you can actually multiply simple matrices.

c) Now solve this matrix equation for the column matrix $\vec{x}$ using the inverse matrix, and then write out the individual scalar solutions of the original system for each individual variable.

d) Check by backsubstitution into the original two equations that your solution is actually a solution.

## ▶ solution

① a) $A = \begin{bmatrix} -3 & -6 & 4 & 7 \\ 2 & 4 & -2 & -2 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} -10 \\ 6 \end{bmatrix}$

$C = \begin{bmatrix} -3 & -6 & 4 & 7 & -10 \\ 2 & 4 & -2 & -2 & 6 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{b)}$

$\begin{bmatrix} 2 & 4 & -2 & -2 & 6 \\ -3 & -6 & 4 & 7 & -10 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & -1 & -1 & 3 \\ -3 & -6 & 4 & 7 & -10 \end{bmatrix}$

$\xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 2 & -1 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \boxed{\begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{bmatrix}}$

c) $\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ L & F & L & F \end{array}$

$x_2 = t_1,\ x_4 = t_2$

$x_1 + 2x_2 + 3x_4 = 2 \rightarrow x_1 = 2 - 2t_1 - 3t_2$

$x_3 + 4x_4 = -1 \rightarrow x_3 = -1 - 4t_2$

$\boxed{x_1 = 2 - 2t_1 - 3t_2,\ x_2 = t_1,\ x_3 = -1 - 4t_2,\ x_4 = t_2}$

d) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t_1 - 3t_2 \\ t_1 \\ -4t_2 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

② a) $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ $\quad$ b) $A^{-1} = \frac{1}{2(3) - 1(4)} \begin{bmatrix} 3 & -1 \\ -4 & 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$

$A^{-1}[A \ \vec{x} = \vec{b}]$

$A^{-1}A = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3(2) - 1(4) & 3(1) - 1(3) \\ -4(2) + 2(4) & -4(1) + 2(3) \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

c) $\vec{x} = A^{-1}\vec{b} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3(-2) - 1(4) \\ -4(-2) + 2(4) \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} -10 \\ 16 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \boxed{x_1 = -5,\ x_2 = 8}$

d) $2(-5) + 1(8) = -2\,?$

$-10 + 8 = -2$ ✓

$4(-5) + 3(8) = 4\,?$

$-20 + 24 = 4$ ✓