MAT2705-03/04 11S FINALEXAM (1)

1 a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -8 & 2 \\ 2 & -5 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ F_{20}\cos 4t \end{bmatrix}}_{\vec{F}}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b) $0 = |A-\lambda I| = \begin{vmatrix} -8-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = (\lambda+8)(\lambda+5)-4$

$= \lambda^2+13\lambda+36 \rightarrow \lambda = \frac{-13\pm\sqrt{13^2-4(36)}}{2} = \frac{-13\pm 5}{2}$

$= \boxed{-4, -9 = \lambda_1, \lambda_2} \quad |\lambda_1|<|\lambda_2|$

$\lambda=-4$:

$A+4I = \begin{bmatrix} -8+4 & 2 \\ 2 & -5+4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ L F

$x_2 = t, \; x_1 = \frac{1}{2}t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2\, t \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$t=2: \quad \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda=-9$:

$A+9I = \begin{bmatrix} -8+9 & 2 \\ 2 & -5+9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ L F

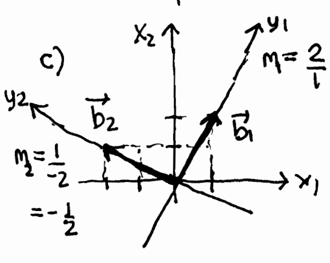
$x_2 = t, \; x_1 = -2t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

$\boxed{B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}$

$B^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad A_B = B^{-1}AB = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix}$

maple agrees.

c)



$m = \frac{2}{1} = 2$

$\boxed{\begin{matrix} m_1 = 2 \\ m_2 = -\frac{1}{2} \end{matrix}}$

d) $\vec{x}'' = A\vec{x}, \quad \vec{x} = B\vec{y}$

$B^{-1}(B\vec{y})'' = B^{-1}ABy \rightarrow \boxed{\vec{y}'' = A_B\vec{y} \implies \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix}}$

$y_1'' = -4y_1, \quad y_1''+4y_1 = 0 \longrightarrow \omega_1 = 2: \; \boxed{y_1 = c_1\cos 2t + c_2\sin 2t}$

$y_2'' = -9y_2 \quad y_2''+9y_2 = 0 \longrightarrow \omega_2 = 3: \; \boxed{y_2 = c_3\cos 3t + c_4\sin 3t}$

$\vec{X} = B\begin{bmatrix} c_1\cos 2t + c_2\sin 2t \\ c_3\cos 3t + c_4\sin 3t \end{bmatrix} \quad \vec{X}(0) = B\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = B^{-1}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\vec{X}' = B\begin{bmatrix} -2c_1\sin 2t + 2c_2\cos 2t \\ -3c_3\sin 3t + 3c_4\cos 3t \end{bmatrix} \quad \vec{X}'(0) = B\begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c_2 = 0 \\ c_4 = 0 \end{matrix}$

$\vec{x} = B\begin{bmatrix} \cos 2t \\ -2\cos 3t \end{bmatrix} = \cos 2t\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\cos 3t\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} \cos 2t + 4\cos 3t \\ 2\cos 2t - 2\cos 3t \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$ agrees with Maple!

e) $B^{-1}\vec{F} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 35\cos 4t = \begin{bmatrix} 14\cos 4t \\ 7\cos 4t \end{bmatrix}$

$\vec{y}'' = A_B\vec{y} + B^{-1}\vec{F}: \quad \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4y_1 + 14\cos 4t \\ -9y_2 + 7\cos 4t \end{bmatrix}$

$y_1''+4y_1 = 14\cos 4t \qquad y_{1p} = c_5\cos 4t + c_6\sin 4t$

$y_2''+9y_2 = 7\cos 4t \qquad y_{2p} = c_7\cos 4t + c_8\sin 4t$

$y_{1p}''+4y_{1p} = (-16+4)(c_5\cos 4t + c_6\sin 4t) = 14\cos 4t$

$-12c_5 = 14, \quad -12c_6 = 0$

$c_5 = -7/6 \qquad y_{1p} = -7/6\cos 4t$

$y_{2p}''+9y_{2p} = (-16+9)(c_7\cos 4t + c_8\sin 4t) = 7\cos 4t$

$-7c_7 = 7, \quad 7c_8 = 0$

$c_7 = -1, \quad y_{2p} = -\cos 4t$

$\boxed{\begin{matrix} y_1 = c_1\cos 2t + c_2\sin 2t - 7/6\cos 4t \\ y_2 = c_3\cos 3t + c_4\sin 3t - \cos 4t \end{matrix}}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B\begin{bmatrix} c_1\cos 2t + c_2\sin 2t - 7/6\cos 4t \\ c_3\cos 3t + c_4\sin 3t - \cos 4t \end{bmatrix}$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = B\begin{bmatrix} -2c_1\sin 2t + 2c_2\cos 2t + 14/3\sin 4t \\ -3c_3\sin 3t + 3c_4\cos 3t + 4\sin 4t \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B\begin{bmatrix} c_1 - 7/6 \\ c_3 - 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} c_1 - 7/6 \\ c_3 - 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

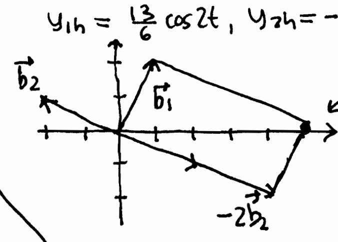
$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = B\begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c_2 = 0 \\ c_4 = 0 \end{matrix} \qquad \downarrow \begin{matrix} c_1 = 1 + 7/6 = 13/6 \\ c_3 = -2 + 1 = -1 \end{matrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B\begin{bmatrix} \frac{13}{6}\cos 2t - \frac{7}{6}\cos 4t \\ -\cos 3t - \cos 4t \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -7 \\ -6 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} +5/6 \\ -10/3 \end{bmatrix}$

(f) $= \underbrace{\frac{13}{6}\cos 2t}_{y_{1h}=}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \underbrace{-\cos 3t}_{y_{2h}=}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \cos 4t\underbrace{\begin{bmatrix} 5/6 \\ -10/3 \end{bmatrix}}_{\vec{b}_3}$

(e) $= \begin{bmatrix} \frac{13}{6}\cos 2t + 2\cos 3t + \frac{5}{6}\cos 4t \\ \frac{13}{3}\cos 2t - \cos 3t - \frac{10}{3}\cos 4t \end{bmatrix}$ agrees with Maple

$y_{1h} = \frac{13}{6}\cos 2t, \quad y_{2h} = -\cos 3t$



$\vec{X}(0) = \langle 5, 0\rangle = \vec{b}_1 - 2\vec{b}_2$

$\langle y_1, y_2\rangle = \langle 1, -2\rangle$ exactly right!

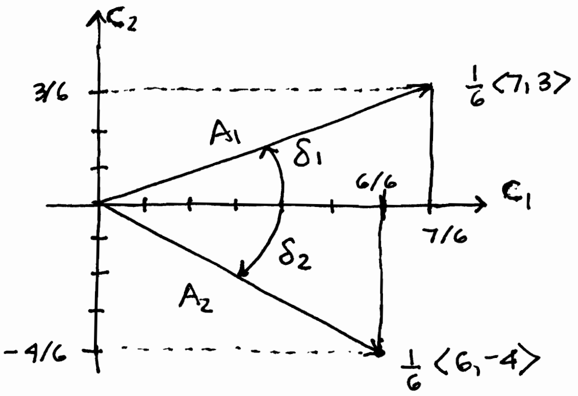
Note:

$\boxed{\begin{matrix} y_1 = \frac{13}{6}\cos 2t - \frac{7}{6}\cos 4t \\ y_2 = -\cos 2t - \cos 4t \end{matrix}}$

b) $y_{1h} = \frac{1}{6}(7\cos 2t + 3\sin 2t) \rightarrow \langle \frac{7}{6}, \frac{3}{6} \rangle$

$y_{2h} = \frac{1}{6}(6\cos 3t - 4\sin 3t) \rightarrow \langle \frac{6}{6}, -\frac{4}{6} \rangle$

$\langle c_1, c_2 \rangle$



$A_1 = \frac{1}{6}\sqrt{7^2+3^2} = \frac{\sqrt{58}}{6} \approx 1.270$

$\delta_1 = \arctan \frac{3}{7} \approx 0.405 \approx 23.2°$ $\boxed{23°}$

$\delta_2 = -\arctan \frac{4}{6} = -\arctan \frac{2}{3} \approx -0.588 \approx -33.7°$ $\boxed{-34°}$

$A_2 = \frac{1}{6}\sqrt{6^2+4^2} = \frac{\sqrt{52}}{6} = \frac{2\sqrt{13}}{6} = \frac{\sqrt{13}}{3} \approx 1.202$

$\delta_1 - \delta_2 = \underbrace{\arctan \frac{3}{7} + \arctan \frac{2}{3}}_{\alpha \; + \; \beta} = \arctan \frac{23}{15} \overset{\approx 0.993}{\approx} 56.9°$ $\boxed{57°}$

$\hookrightarrow \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{3}{7}+\frac{2}{3}}{1-\frac{3}{7}\cdot\frac{2}{3}} = \frac{23/21}{5/7} = \frac{23}{15}$ tangent sum formula

$\delta_1 - \delta_2 = \alpha + \beta = \arctan \frac{23}{15}$ ✓

$\frac{A_1}{A_2} = \frac{\sqrt{58}/6}{\sqrt{13}/3} = \frac{1}{2}\sqrt{\frac{58}{13}} = \frac{\sqrt{58}}{26} \approx 1.056$ (nearly same amplitudes — about 1/6 cycle out of phase)