MAT2705-03/04 11S TEST2 Answers

① a) From diagram $\langle 4,4\rangle = -2\vec{v}_1 + 2\vec{v}_2$

so $\boxed{\langle y_1,y_2\rangle = \langle -2,2\rangle}$

(see diagram on next page)

b) $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{v}_3$

$$\begin{bmatrix}-1 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}4\\ 4\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 1 & 3\end{bmatrix}^{-1}\begin{bmatrix}4\\ 4\end{bmatrix} = \frac{1}{-3-1}\begin{bmatrix}3 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}4\\ 4\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}-3 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}4\\ 4\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-3(4)+4\\ 4+4\end{bmatrix} = \begin{bmatrix}-2\\ 2\end{bmatrix}$$

so $\boxed{\vec{v}_3 = -2\vec{v}_1 + 2\vec{v}_2}$

c) $$= -2\begin{bmatrix}-1\\ 1\end{bmatrix} + 2\begin{bmatrix}1\\ 3\end{bmatrix} = \begin{bmatrix}2+2\\ -2+6\end{bmatrix} = \begin{bmatrix}4\\ 4\end{bmatrix} \checkmark$$

d) yes, they agree!

② a) $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{v}_4$

$$\begin{bmatrix}2 & 3 & 7\\ -1 & 4 & 2\\ 3 & -1 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1\\ 5\\ -4\end{bmatrix}$$

$$\begin{bmatrix}2 & 3 & 7 & 1\\ -1 & 4 & 2 & 5\\ 3 & -1 & 5 & -4\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 2 & -1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$x_1\ x_2\ x_3$
L L F

$x_1 = -2t + -1$
$x_2 = -t + 1$
$x_3 = t$

$\boxed{\text{yes}}$, it can be expressed in terms of these 3 vectors:

$$\begin{bmatrix}1\\ 5\\ -4\end{bmatrix} = (-2t-1)\begin{bmatrix}2\\ -1\\ 3\end{bmatrix} + (1-t)\begin{bmatrix}3\\ 4\\ -1\end{bmatrix} + t\begin{bmatrix}7\\ 2\\ 5\end{bmatrix}$$

reverse sign:

b) $$\begin{bmatrix}2 & 3 & 7 & 1\\ -1 & 4 & 2 & 5\\ 3 & -1 & 5 & 4\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \to 0 = 1$$

$x_1\ x_2\ x_3$
L L F

inconsistent system: no soln.

$\boxed{\text{no}}$, $\vec{v}_4$ cannot be expressed in terms of these vectors.

c) Since there is one free variable, the homogeneous linear system admits nonzero solutions, so the vectors are not linearly independent. Indeed one independent linear relationship exists among the 3 vectors.

③ a) $A\vec{x} = \vec{0}$:

$$\begin{bmatrix}1 & 2 & -1 & -5\\ 5 & -1 & 6 & 8\\ 3 & 4 & -1 & -9\\ 2 & 1 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & -1 & -5 & 0\\ 5 & -1 & 6 & 8 & 0\\ 3 & 4 & -1 & -9 & 0\\ 2 & 1 & 1 & -1 & 0\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 1 & 1 & 0\\ 0 & 1 & -1 & -3 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$x_1\ x_2\ x_3\ x_4$
L L F F

$x_3 = t_1$, $x_4 = t_2$
$x_1 = -t_1 - t_2$
$x_2 = t_1 + 3t_2$

$$\boxed{\vec{x} = \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}-t_1 - t_2\\ t_1 + 3t_2\\ t_1\\ t_2\end{bmatrix}} = t_1\begin{bmatrix}-1\\ 1\\ 1\\ 0\end{bmatrix} + t_2\begin{bmatrix}-1\\ 3\\ 0\\ 1\end{bmatrix}$$

b) $\boxed{\vec{u}_1 = \langle -1,1,1,0\rangle,\ \vec{u}_2 = \langle -1,3,0,1\rangle}$

c) $\vec{u}_1$: $\boxed{-\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}}$

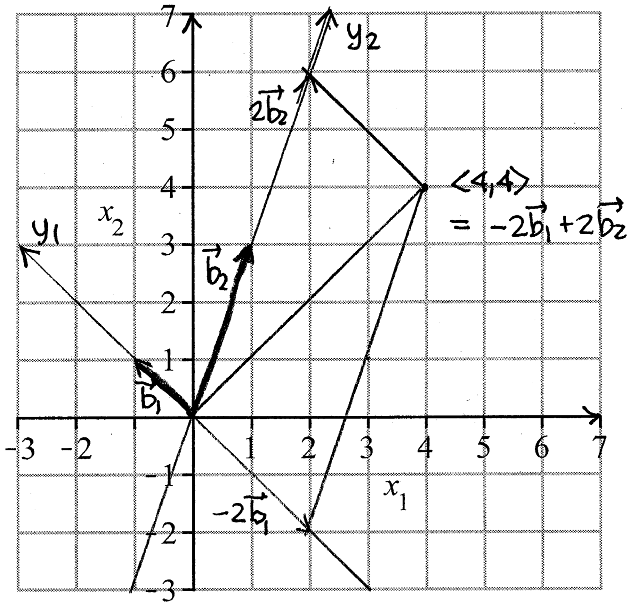
$\vec{u}_2$: $\boxed{-\vec{v}_1 + 3\vec{v}_2 + \vec{v}_4 = \vec{0}}$

d) This set only has $4-2=2$ linearly independent vectors, so it represents a $\boxed{\text{plane}}$ through the origin of $\mathbb{R}^4$.

Any 2 of the 4 vectors will form a basis of the plane since clearly no two are proportional, but our algorithm picks out the first two: $\{\vec{v}_1, \vec{v}_2\}$

(since $\vec{v}_3$ and $\vec{v}_4$ are linear combinations of them by the above relationships)

Don't confuse $\vec{u}_1, \vec{u}_2$ with vectors in the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ or its span — they are instead coefficient vectors of linear combinations of these 4 vectors.



y_2
$2\vec{b}_2$
$\langle 4,4\rangle$
$= -2\vec{b}_1 + 2\vec{b}_2$
y_1
x_2
$\vec{b}_2$
$\vec{b}_1$
x_1
$-2\vec{b}_1$