MAT 2705-03/04 11F Q8 ANSWER KEY

a) $\begin{bmatrix}x_1'\\x_2'\\x_3'\end{bmatrix} = \underbrace{\begin{bmatrix}-2&0&0\\7&-9&-7\\0&0&-2\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \quad \begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix}=\begin{bmatrix}2\\-2\\1\end{bmatrix}$

b) Maple gives in random order

$\lambda_1=-2$, $\lambda_2=-2$, $\lambda_3=-9$ $\quad B=\begin{bmatrix}1&1&0\\0&1&1\\1&0&0\end{bmatrix}$ $\quad$ $\vec{b}_1$ & $\vec{b}_2$ can also be exchanged in order.

c) $AB = \langle A\vec{b}_1 | A\vec{b}_2 | A\vec{b}_3\rangle$ (Maple)

$= \begin{bmatrix}-2&-2&0\\0&-2&-9\\-2&0&0\end{bmatrix}$ all agree:

$-2\vec{b}_1 \quad -2\vec{b}_2 \quad -9\vec{b}_3$

d) $B^{-1}=\begin{bmatrix}0&0&1\\1&0&-1\\-1&1&1\end{bmatrix}, \quad B^{-1}AB=\begin{bmatrix}-2&0&0\\0&-2&0\\0&0&-9\end{bmatrix}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}$ ✓

e) $\vec{x}(0)=B\vec{y}(0) \to \vec{y}(0)=B^{-1}x(0)=\begin{bmatrix}0&0&1\\1&0&-1\\-1&1&1\end{bmatrix}\begin{bmatrix}2\\-2\\1\end{bmatrix}=\begin{bmatrix}1\\1\\-3\end{bmatrix}$

f) $\vec{x}'=A\vec{x} \xrightarrow{\vec{x}=B\vec{y}} B^{-1}[B\vec{y}'=ABy]$

$\vec{y}'=B^{-1}AB\vec{y}=A_B\vec{y}$

$\begin{bmatrix}y_1'\\y_2'\\y_3'\end{bmatrix}=\begin{bmatrix}-2&0&0\\0&-2&0\\0&0&-9\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}-2y_1\\-2y_2\\-9y_3\end{bmatrix}$ $\quad y_i \neq y_i(0)$ — function, single value of function

many of you replaced $\vec{y}$ by $\vec{y}(0)$ here.

$y_1'=-2y_1 \quad y_1=c_1e^{-2t}$

$y_2'=-2y_2 \quad y_2=c_2e^{-2t}$

$y_3'=-9y_3 \quad y_3=c_3e^{-9t}$

many of you "solved" the DE by writing $y_i=\frac{1}{\lambda_i}y_i'$!!

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&1&0\\0&1&1\\1&0&0\end{bmatrix}\begin{bmatrix}c_1e^{-2t}\\c_2e^{-2t}\\c_3e^{-9t}\end{bmatrix}$

g) The initial conditions are $\vec{x}(0)=B\vec{C}$

$\vec{C}=B^{-1}\vec{x}(0)=\vec{y}(0)=\langle 1,1,-3\rangle$ above

h) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&1&0\\0&1&1\\1&0&0\end{bmatrix}\begin{bmatrix}e^{-2t}\\e^{-2t}\\-3e^{-9t}\end{bmatrix}=\begin{bmatrix}e^{-2t}+e^{-2t}\\e^{-2t}-3e^{-9t}\\e^{-2t}\end{bmatrix}$

$=\begin{bmatrix}2e^{-2t}\\e^{-2t}-3e^{-9t}\\e^{-2t}\end{bmatrix}$ $\quad$ matrix form → scalar form

$x_1=2e^{-2t}$,
$x_2=e^{-2t}-3e^{-9t}$
$x_3=-3e^{-9t}$

$=\underbrace{e^{-2t}\begin{bmatrix}1\\0\\1\end{bmatrix}+e^{-2t}\begin{bmatrix}1\\1\\0\end{bmatrix}}-3e^{-9t}\begin{bmatrix}0\\1\\0\end{bmatrix}$

$= e^{-2t}\underbrace{\begin{bmatrix}2\\1\\1\end{bmatrix}}_{\vec{v}_1} + -3e^{-9t}\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\vec{v}_2}$

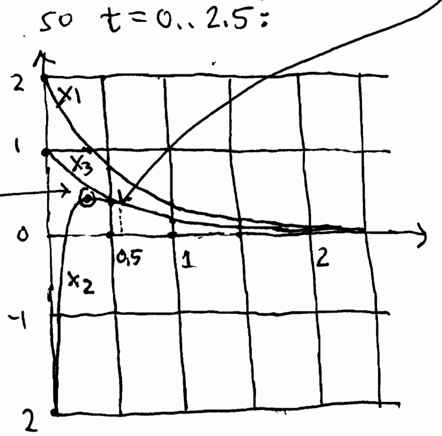
h) soln confined to plane of $\vec{v}_1,\vec{v}_2$: basis $=\{\vec{v}_1,\vec{v}_2\}$

i) $\tau_1=\frac{1}{2}, \quad \tau_2=\frac{1}{9} \qquad \tau_1>\tau_2$

$\approx 0.5 \qquad \approx 0.11 \qquad 5\tau_2\approx .55$

so $t=0..2.5$:

j)



merge x_2, x_3 at this t-value, right!

$x_2-x_3\sim-3e^{-9t}$

global max of x_2

k) $x_2=e^{-2t}-3e^{-9t}$

$x_2'=[-2e^{-2t}+27e^{-9t}=0]\,e^{9t}$

$-2e^{7t}+27=0, \quad e^{7t}=\frac{27}{2},$

$\boxed{t=\frac{1}{7}\ln\left(\frac{27}{2}\right)\approx 0.372}$

$x_2= e^{-\frac{2}{7}\ln\left(\frac{27}{2}\right)}-3e^{-\frac{9}{7}\ln\frac{27}{2}}$

$=\left(\frac{2}{27}\right)^{2/7}-3\left(\frac{2}{27}\right)^{9/7}=\left(\frac{2}{27}\right)^{2/7}\left(1-3\cdot\underbrace{\frac{2}{27}}\right)$ $\quad 2/9$

$=\boxed{\frac{7}{9}\left(\frac{2}{27}\right)^{2/7}\approx 0.370}$