MAT2705-02/06 10F Quiz 8 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. $y''+\frac{1}{2}y'+2y=\cos(2t)$ a) Identify the values of the damping constant $k_0=1/\tau_0$, the natural frequency ω_0, and the quality factor $Q=\omega_0\tau_0$.

b) Use the method of undetermined coefficients to find the steady state sinusoidal solution (the particular solution y_p) of this driven harmonic oscillator DE.

c) Evaluate the amplitude and phase shift of this steady state solution.

d) Now find the steady state solution for a general positive frequency $y''+\frac{1}{2}y'+2y=\cos(\omega t)$, $\omega>0$.

e) Evaluate the amplitude $A(\omega)$ of this sinusoidal function. Show that it simplifies to

$A(\omega)=2\left(16-15\omega^2+4\omega^4\right)^{-\frac{1}{2}}$. Does $A(2)$ reduce to your result for part c) as it should? Evaluate $A(0)$.

f) Use calculus to show that the peak amplitude occurs exactly at the value $\omega_{peak}=\sqrt{15/8}$, for which $A(\omega_{peak})=\frac{8}{\sqrt{31}}$ (no need to verify this fact). How does the ratio $A(\omega_{peak})/A(0)$ compare to the Q value?

► solution

1) a) $y''+\underbrace{\left(\tfrac{1}{2}\right)}_{=k_0}y'+\underbrace{(2)}_{=\omega_0^2}y=\cos 2t \rightarrow \boxed{k_0=\tfrac{1}{2},\ \tau_0=2,\ \omega_0=\sqrt{2}\approx 1.414,\ Q=2\sqrt{2}\approx 2.828}$

b)
$2\ [y_p=c_1\cos 2t+c_2\sin 2t$

$\tfrac{1}{2}\ [y_p'=-2c_1\sin 2t+2c_2\cos 2t$

$1\ [y_p''=-4c_1\cos 2t-4c_2\sin 2t$

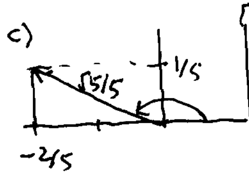
$y_p''+\tfrac{1}{2}y_p'+2y_p=[(2-4)c_1+c_2]\cos 2t=\cos 2t$
$\qquad+[-c_1+(2-4)c_2]\sin 2t$

$-2c_1+c_2=1$
$-c_1-2c_2=0$
$\begin{bmatrix}-2&1\\-1&-2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\frac{1}{4+1}\begin{bmatrix}-2&-1\\1&-2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{5}\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}-2/5\\1/5\end{bmatrix}$

$\boxed{y_p=\tfrac{1}{5}[-2\cos 2t+\sin 2t]}$

c)



$\boxed{\begin{aligned}&A=\tfrac{1}{5}\sqrt{(-2)^2+1^2}=\tfrac{\sqrt{5}}{5}=\tfrac{1}{\sqrt{5}}\\&\delta=\pi-\arctan(1/2)\end{aligned}}$

d)
$2\ [y_p=c_1\cos\omega t+c_2\sin\omega t$

$\tfrac{1}{2}\ [y_p'=-\omega c_1\sin\omega t+\omega c_2\cos\omega t$

$1\ [y_p''=-\omega^2c_1\cos\omega t-\omega^2c_2\sin\omega t$

$y_p''+\tfrac{1}{2}y_p'+2y_p=[(2-\omega^2)c_1+\tfrac{1}{2}\omega c_2]\cos\omega t=\cos\omega t$
$\qquad+[-\tfrac{1}{2}\omega c_1+(2-\omega^2)c_2]\sin\omega t$

$(2-\omega^2)c_1+\tfrac{1}{2}\omega c_2=1$
$-\tfrac{1}{2}\omega c_1+(2-\omega^2)c_2=0$
$\begin{bmatrix}2-\omega^2&\tfrac{1}{2}\omega\\-\tfrac{1}{2}\omega&2-\omega^2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\frac{1}{(2-\omega^2)^2+\frac{1}{4}\omega^2}\begin{bmatrix}2-\omega^2&-\tfrac{1}{2}\omega\\\tfrac{1}{2}\omega&2-\omega^2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{(2-\omega^2)^2+\frac{1}{4}\omega^2}\begin{bmatrix}2-\omega^2\\\tfrac{1}{2}\omega\end{bmatrix}$

$\boxed{y_p=\frac{(2-\omega^2)\cos\omega t+\frac{1}{2}\omega\sin\omega t}{(2-\omega^2)^2+\frac{1}{4}\omega^2}}$

e) $A(\omega)=\frac{\sqrt{(2-\omega^2)^2+\frac{1}{4}\omega^2}}{(2-\omega^2)^2+\frac{1}{4}\omega^2}=\frac{1}{\sqrt{(2-\omega^2)^2+\frac{1}{4}\omega^2}}$

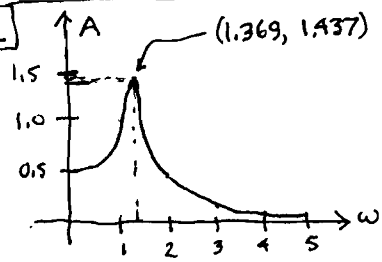
$=(4-4\omega^2+\omega^4+\tfrac{1}{4}\omega^2)^{-1/2}=(4-\tfrac{15}{4}\omega^2+\omega^4)^{-1/2}$

$=\left(\frac{16-15\omega^2+4\omega^4}{4}\right)^{-1/2}=\boxed{2(16-15\omega^2+4\omega^4)^{-1/2}}$

$\left(\tfrac{1}{4}\right)^{-1/2}$

$A(2)=2(16-60+64)^{-1/2}=\frac{2}{\sqrt{20}}=\frac{1}{\sqrt{5}}$ ✓ same.

$A(0)=2(16)^{-1/2}=2/4=\boxed{1/2}$



f) $0=A'(\omega)=2(-\tfrac{1}{2})(\cdots)^{-3/2}(-30\omega+16\omega^3)$

$2\omega(8\omega^2-15)=0$

$\omega^2=15/8,\ \boxed{\begin{aligned}&\omega=\sqrt{15/8}\\&\approx 1.3693\end{aligned}}$

$\frac{A(\omega_{peak})}{A(0)}=\frac{8/\sqrt{31}}{1/2}=16\sqrt{31}\approx 2.874$

$Q\approx 2.828$

pretty close for rule of thumb guide.

yields nice resonance peak:

[Note $\omega\approx 1.3693$ is a bit smaller than $\omega_0\approx 1.414$]