MAT2705-02/06 10F Final Exam Answers (1)

① a) $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -12x_1+2x_2 \\ 16x_1-8x_2+70\sin 3t \end{bmatrix} = \underbrace{\begin{bmatrix} -12 & 2 \\ 16 & -8 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 70\sin 3t \end{bmatrix}}_{\vec f}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix}$

b) $0 = |A-\lambda I| = \begin{vmatrix} -12-\lambda & 2 \\ 16 & -8-\lambda \end{vmatrix} = (\lambda+12)(\lambda+8)-32 = \lambda^2+20\lambda+\overbrace{96-32}^{64} \to \lambda = -4, -16$

$\lambda=-4$: $A+4I = \begin{bmatrix} -8 & 2 \\ 16 & -4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (L F) $x_2=t$ $x_1 = \frac{1}{4}x_2 = \frac{1}{4}t$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/4\, t \\ t \end{bmatrix} = t\begin{bmatrix} 1/4 \\ 1 \end{bmatrix} \xrightarrow{\times 4} \vec b_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\lambda=-16$: $A+16I = \begin{bmatrix} 4 & 2 \\ 16 & 8 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (L F) $x_2=t$ $x_1 = -\frac{1}{2}x_2 = -\frac{1}{2}t$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \xrightarrow{\times 2} \vec b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$B = \langle \vec b_1 | \vec b_2 \rangle = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}$, $B^{-1} = \frac{1}{2+4}\begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix}$, $B^{-1}AB = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix} = A_B$ scale up for smallest integer eigenvectors

c) $\vec b_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \to m_1 = \frac{4}{1} = 4$, $\vec b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $m_2 = \frac{2}{-1} = -2$.

unscaled basis $B_0 = \begin{bmatrix} 1/4 & -1/2 \\ 1 & 1 \end{bmatrix}$, $B_0^{-1} = \frac{4}{3}\begin{bmatrix} 1 & 1/2 \\ -1 & 1/4 \end{bmatrix}$

d) $B^{-1}f = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 70\sin 3t \end{bmatrix} = \frac{70}{6}\sin 3t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{35}{3}\sin 3t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$B_0^{-1}\vec f = \langle \frac{140}{3}, \frac{70}{3} \rangle \sin 3t$

$\vec y_p \to \langle 4(-\frac{7}{3}), 2(\frac{5}{3}) \rangle \sin 3t = \langle -\frac{28}{3}, \frac{10}{3} \rangle \sin 3t$

$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{35}{3}\sin 3t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to$ $y_1''+4y_1 = \frac{35}{3}\sin 3t$, $y_2''+16y_2 = \frac{35}{3}\sin 3t$.

$y_{1h} = c_1\cos 2t + c_2\sin 2t$
$y_{2h} = c_3\cos 4t + c_4\sin 4t$

$4\,[y_{1p} = c_5\sin 3t]$
$1\,[y_{1p}'' = -9c_5\sin 3t]$

$y_{1p}''+4y_{1p} = \underbrace{(4-9)}_{-5}c_5\sin 3t = \frac{35}{3}\sin 3t$

$c_5 = -\frac{7}{3}$ $\quad y_{1p} = -\frac{7}{3}\sin 3t$

$16\,[y_{2p} = c_6\sin 3t]$
$1\,[y_{2p}'' = -9c_6\sin 3t]$

$y_{2p}''+16y_{2p} = \underbrace{(16-9)}_{7}c_6\sin 3t = \frac{35}{3}\sin 3t$

$c_6 = \frac{5}{3}$ $\quad y_{2p} = \frac{5}{3}\sin 3t$

$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boxed{y_1 = y_{1h}+y_{1p} = c_1\cos 2t + c_2\sin 2t - 7/3\sin 3t}$, $\boxed{y_2 = y_{2h}+y_{2p} = c_3\cos 4t + c_5\sin 4t + 5/3\sin 3t}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} c_1\cos 2t + c_2\sin 2t - 7/3\sin 3t \\ c_3\cos 4t + c_4\sin 4t + 5/3\sin 3t \end{bmatrix}$ $\quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -2c_1\sin 2t + 2c_2\cos 2t - 7\cos 3t \\ -4c_3\sin 4t + 4c_4\cos 4t + 5\cos 3t \end{bmatrix}$ $\quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}}_{\text{invertible}}\underbrace{\begin{bmatrix} 2c_2-7 \\ 4c_4+5 \end{bmatrix}}_{\therefore\ = \begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to$ $c_2 = 7/2$, $c_4 = -5/4$.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} \overbrace{\cos 2t + \frac{7}{2}\sin 2t}^{y_{1h}} - \frac{7}{3}\sin 3t \\ \underbrace{-\cos 4t - \frac{5}{4}\sin 4t}_{y_{2h}} + \frac{5}{3}\sin 3t \end{bmatrix} = \begin{bmatrix} (\cos 2t + \frac{7}{2}\sin 2t - \frac{7}{3}\sin 3t) - (-\cos 4t - \frac{5}{4}\sin 4t + \frac{5}{3}\sin 3t) \\ 4(\cos 2t + \frac{7}{2}\sin 2t - \frac{7}{3}\sin 3t) + 2(-\cos 4t - \frac{5}{4}\sin 4t + \frac{5}{3}\sin 3t) \end{bmatrix}$

$= \begin{bmatrix} \cos 2t + \frac{7}{2}\sin 2t + \cos 4t + \frac{5}{4}\sin 4t + \overbrace{(-\frac{7}{3}-\frac{5}{3})}^{-4}\sin 3t \\ 4\cos 2t + 14\sin 2t - 2\cos 4t - \frac{5}{2}\sin 4t + \overbrace{(-\frac{28}{3}+\frac{10}{3})}^{-6}\sin 3t \end{bmatrix}$ agrees with Maple

$\boxed{\begin{aligned} x_1 &= \cos 2t + \tfrac{7}{2}\sin 2t + \cos 4t + \tfrac{5}{4}\sin 4t - 4\sin 3t \\ x_2 &= 4\cos 2t + 14\sin 2t - 2\cos 4t - \tfrac{5}{2}\sin 4t - 6\sin 3t \end{aligned}}$

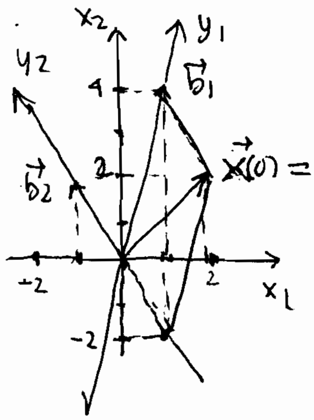
e) $\boxed{\begin{aligned} y_{1h} &= \cos 2t + \tfrac{7}{2}\sin 2t \\ y_{2h} &= -\cos 4t - \tfrac{5}{4}\sin 4t \end{aligned}}$ $\to \times 4$, $\to \times 2$ unscaled basis

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\cos 2t + \frac{7}{2}\sin 2t)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-\cos 4t - \frac{5}{4}\sin 4t)\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \sin 3t\underbrace{\begin{bmatrix} -4 \\ -6 \end{bmatrix}}_{\vec b_3}$

f) $\begin{bmatrix} x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} \to \begin{bmatrix} y_1(0)\\ y_2(0)\end{bmatrix} = B^{-1}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \frac{1}{6}\begin{bmatrix}2&1\\-4&1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \frac{2}{6}\begin{bmatrix}2+1\\-4+1\end{bmatrix} = \underbrace{\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1\\c_3\end{bmatrix}}$

already evaluated above since $\vec{y}_p(0) = \vec{0}$ does not contribute

$\vec{x}(0) = 1\vec{b}_1 - \vec{b}_2$



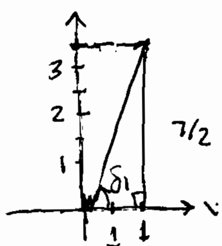
scaled basis vectors make plot simpler

$\vec{x}(0) = \vec{b}_1 - \vec{b}_2 \to \vec{y}(0) = \langle 1,-1\rangle$

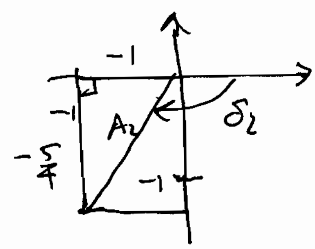
$\llcorner\ 4(\frac{1}{4}\vec{b}_1) - 2(\frac{1}{2}\vec{b}_2) \longrightarrow \vec{y}(0) \to \langle 4,-2\rangle$ unscaled basis

g) $y_{1h} = \cos 2t + \frac{7}{2}\cos 2t$

$y_{2h} = -\cos 4t - \frac{5}{4}\sin 4t$



1st quad



3rd quad

$A_1 = \sqrt{(1)^2 + (\frac{7}{2})^2} = \sqrt{1 + \frac{49}{4}} = \frac{\sqrt{53}}{2} \approx 3.64$

$\delta_1 = \arctan\frac{7}{2} \approx 74°$

$A_2 = \sqrt{(-1)^2 + (-\frac{5}{4})^2} = \sqrt{1 + \frac{25}{16}} = \frac{\sqrt{41}}{4}$

≈ 1.60

$\delta_2 = -\pi + \arctan(\frac{5}{4}) \approx -129°$