MAT2500-03/04 09S Test 1

a) $\vec{r}(t) = \langle t^2, 1-t^3, t^3 \rangle$

$\vec{r}\,'(t) = \langle 2t, -3t^2, 3t^2 \rangle$

$= t \langle 2, -3t, 3t \rangle$

$\vec{r}\,''(t) = \langle 2, -6t, 6t \rangle = \vec{a}(t)$

$|\vec{r}\,'(t)| = |t| \sqrt{4+9t^2+9t^2}$

$= |t| \sqrt{4+18t^2}$

$= \sqrt{4t^2+18t^4}$

$\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle 2t, -3t^2, 3t^2 \rangle}{\sqrt{4t^2+18t^4}}$

$= \underbrace{\dfrac{t}{|t|}}_{\text{"sgn}t\text{"}} \dfrac{\langle 2, -3t, 3t \rangle}{\sqrt{4+18t^2}}$

b) $\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -3t^2 & 3t^2 \\ 2 & -6t & 6t \end{vmatrix}$

$= \langle -18t^3+18t^3, 6t^2-12t^2, -12t^2+6t^2 \rangle$

$= \langle 0, -6t^2, -6t^2 \rangle = 6t^2 \langle 0, -1, -1 \rangle$

$\hat{B}(t) = \widehat{\vec{r}\,'(t) \times \vec{r}\,''(t)} = \boxed{\dfrac{\langle 0, -1, -1 \rangle}{\sqrt{2}}}$ constant!

c) let $\vec{n} = \langle 0, 1, 1 \rangle = -\sqrt{2}\,\hat{B}$

$\vec{r}_0 = \vec{r}(0) = \langle 0, 1, 0 \rangle$

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$\langle 0, 1, 1 \rangle \cdot \langle x-0, y-1, z-0 \rangle = 0$

$y - 1 + z = 0 \qquad \boxed{y + z = 1}$

d) $\hat{N}(t) = \hat{B} \times \hat{T}(t) = \dfrac{1}{\sqrt{2}\sqrt{4t^2+18t^4}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & -1 \\ 2t & -3t^2 & 3t^2 \end{vmatrix}$

$= \dfrac{1}{\sqrt{2}\sqrt{4t^2+18t^4}} \langle -3t^2-3t^2, -2t-0, 0+2t \rangle$

$= \boxed{\dfrac{\langle -6t^2, -2t, 2t \rangle}{\sqrt{2}\sqrt{4t^2+18t^4}}} = \dfrac{2t}{\sqrt{2}|t|} \dfrac{\langle -3t, -1, 1 \rangle}{\sqrt{4+18t^2}}$

$= \boxed{\sqrt{2} \dfrac{t}{|t|} \dfrac{\langle -3t, -1, 1 \rangle}{\sqrt{4+18t^2}}}$

$\hat{N}(t) \cdot \hat{T}(t) = \dfrac{\langle -6t^2, -2t, 2t \rangle}{\sqrt{2}\sqrt{4t^2+18t^4}} \cdot \dfrac{\langle 2t, -3t^2, 3t^2 \rangle}{\sqrt{4t^2+18t^4}}$

$= \dfrac{-12t^3+6t^3+6t^3}{\sqrt{2}(4t^2+18t^4)} = 0$ ✓

e) $s = \displaystyle\int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 \sqrt{4t^2+18t^4}\,dt$

$= \displaystyle\int_0^1 t \sqrt{4+18t^2}\,dt$ (since $t \geq 0$)

$u = 4+18t^2$

$du = 36t\,dt$

$= \displaystyle\int_{t=0}^{t=1} u^{1/2} \frac{du}{36} = \frac{u^{3/2}}{3/2} \frac{1}{36} \Big|_{t=0}^{t=1} = \frac{(4+18t^2)^{3/2}}{54} \Big|_0^1$

$= \dfrac{22^{3/2} - 4^{3/2}}{54} = \boxed{\dfrac{22\sqrt{22}-8}{54}} \approx 1.7628$

$= \dfrac{11}{27}\sqrt{22} - \dfrac{4}{27}$ simplified Maple integral ✓

$\vec{r}(1) - \vec{r}(0) = \langle 1, 0, 1 \rangle - \langle 0, 1, 0 \rangle = \langle 1, -1, 1 \rangle$

$|\vec{r}(1) - \vec{r}(0)| = \boxed{\sqrt{3}} \approx 1.732$

The curve is a bit longer than the straight line as it should be.

f) $a_T = \hat{T}(1) \cdot \vec{a}(1) = \dfrac{\langle 2, -3, 3 \rangle}{\sqrt{22}} \cdot \langle 2, -6, 6 \rangle$

$= \dfrac{4+18+18}{\sqrt{22}} = \dfrac{40}{\sqrt{22}} \approx 8.528$

$a_N = \hat{N}(1) \cdot \vec{a}(1) = \dfrac{\langle -6, -2, 2 \rangle}{\sqrt{2}\sqrt{22}} \cdot \langle 2, -6, 6 \rangle$

$= \dfrac{-12+12+12}{\sqrt{2}\sqrt{22}} = \dfrac{12}{\sqrt{2}\sqrt{22}} = \dfrac{6}{\sqrt{11}}$

≈ 1.809

$a_T^2 + a_N^2 = \dfrac{40^2}{22} + \dfrac{144}{2 \cdot 22} = \dfrac{1600+72}{22}$

$= \dfrac{836}{11} = 76$

$|\vec{a}(1)|^2 = 4+36+36 = 76$ ✓

g) $v'(1) = a_T(1) = \dfrac{40}{\sqrt{22}} > 0$

positive so speed is increasing.