MAT2705-02/06 09F Quiz 9 Print Name (Last, First) ______________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. $x_1'(t) = x_1(t) - 2x_2(t)$, $x_2'(t) = -2x_1(t) + x_2(t)$, $x_3'(t) = 3x_3(t)$, $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 3$.

a) Rewrite this system of DEs **and** its initial conditions in matrix form for the vector variable $\vec{x} = \langle x_1, x_2, x_3 \rangle$ as a column matrix, identifying the coefficient matrix A.

b) Using Maple write down the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (ordered by increasing value) and corresponding matrix of eigenvectors $B = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle$ that it provides you, reordering them if necessary to order them as requested.

c) By hand showing all steps (you should use technology to evaluate the necessary determinant and solve), show that the characteristic equation for the eigenvalues has roots -1 and 3.

d) For each eigenvalue, by hand find a basis of the corresponding eigenspace, collecting your results into a new matrix B and compare your result with Maple's. Do they agree once reordered as above? If not, are they equivalent modulo permutations or rescalings?

e) Use technology to evaluate and write down the inverse matrix B^{-1} and use Maple to evaluate the matrix product $A_D = B^{-1} A B$. Does it evaluate correctly to the diagonalized matrix with the eigenvalues in the correct order?

f) Given that $\vec{x} = B\vec{y}$, if $\vec{x}(0) = \langle 6, 2, 3 \rangle$, find $\vec{y}(0)$. Explain how you did this.

▶ solution

1 a) $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1x_1 - 2x_2 + 0x_3 \\ -2x_1 + 1x_2 + 0x_3 \\ 0x_1 + 0x_2 + 3x_3 \end{bmatrix}$

b) $\lambda = -1, 3, 3$

$B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Maple randomly orders eigenvalues in output, by chance for me they came out ordered by increasing value initially

c) $(A - \lambda I)\vec{x} = 0 \rightarrow (\vec{x} \neq 0)$

$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$\overset{\text{Maple}}{=} -\lambda^3 + 5\lambda^2 - 3\lambda - 9 \overset{\text{factor}}{=} -(\lambda+1)(\lambda-3)^2 = 0$

$\rightarrow \lambda = -1, 3, 3$

d) $\lambda = -1$: $A + I = \begin{bmatrix} 1+1 & -2 & 0 \\ -2 & 1+1 & 0 \\ 0 & 0 & 3+1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\xrightarrow{\text{RREF}}$ $\begin{array}{ccc} L & F & L \end{array}$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $x_1 = x_2 = t$, $x_3 = 0$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}_1}$

$x_2 = t$

d) $\lambda = 3$: $A - 3I = \begin{bmatrix} 1-3 & -2 & 0 \\ -2 & 1-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\text{RREF}}$ $\begin{array}{ccc} L & F & F \end{array}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $x_1 = -x_2 = -t_1$

$x_2 = t_1$, $x_3 = t_2$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_1 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}_2} + t_2 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{b}_3}$

$\langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ last 2 interchanged by Maple (it introduces parameters from right to left)

e) If we use Maples B:

$B^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}$ then $B^{-1} A B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

as it should.

f) $\boxed{\vec{y}(0) = B^{-1}\vec{x}(0) = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+1 \\ 3 \\ -3+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}}$