MAT2705-02/06 09F Quiz 8 Print Name (Last, First) ____________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. $y''+2y'+10y=\cos(3t)$

a) Use the method of undetermined coefficients to find the steady state sinusoidal solution (the particular solution y_p) of this driven harmonic oscillator DE.

b) Evaluate the amplitude and phase shift of this steady state solution.

c) Now consider a general positive frequency $y''+2y'+10y=\cos(\omega t)$, $\omega>0$. Find the steady state solution again.

d) Evaluate the amplitude $A(\omega)$ of this sinusoidal function. Show that it simplifies to the formula

$$A(\omega)=\left(100-16\,\omega^2+\omega^4\right)^{-\frac{1}{2}}.$$

e) Use calculus to show that the peak amplitude occurs at the value $\omega\approx 2.828$. Evaluate $A(0)$, $A(3)$ numerically.

OPTIONAL

f) Use technology to plot $A(\omega)$ for $\omega=0..20$. Sketch what you see, labeling the axes, tickmarks and the vertical intercept and the points on the graph for which $\omega=2.828, 3$.

▶ solution

1) a) $10\,[y_p = c_1\cos 3t + c_2\sin 3t]$

$2\,[y_p' = -3c_1\sin 3t + 3c_2\cos 3t]$

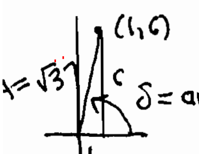
$1\,[y_p'' = -9c_1\cos 3t - 9c_2\sin 3t]$

$y_p''+2y_p'+10y_p = [(10-9)c_1+6c_2]\cos 3t + [-6c_1+(10-9)c_2]\sin 3t = \cos 3t \rightarrow$

$c_1+6c_2=1$
$-6c_1+c_2=0$
$\begin{bmatrix}1&6\\-6&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1&6\\-6&1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{1+36}\begin{bmatrix}1&-6\\6&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1/37\\6/37\end{bmatrix}$

$\boxed{y_p=\frac{1}{37}[\cos 3t+6\sin 3t]=y_{ss}}$



$\delta=\arctan 6$

$\boxed{\begin{matrix}A=\sqrt{37}/37\\ \delta=\arctan 6\end{matrix}}=\frac{1}{\sqrt{37}}$

c) $10\,[y_p = c_1\cos\omega t + c_2\sin\omega t]$

$2\,[y_p' = -\omega c_1\sin 3t + \omega c_2\cos 3t]$

$1\,[y_p'' = -\omega^2 c_1\cos 3t - \omega^2 c_2\sin 3t]$

$y_p''+2y_p'+10y_p = [(10-\omega^2)c_1+2\omega c_2]\cos\omega t + [-2\omega c_1+(10-\omega^2)c_2]\sin\omega t = \cos\omega t \rightarrow$

$\begin{bmatrix}10-\omega^2&2\omega\\-2\omega&10-\omega^2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}\ \end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{(10-\omega^2)^2+4\omega^2}\begin{bmatrix}10-\omega^2&-2\omega\\2\omega&10-\omega^2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$

$=\frac{1}{(10-\omega^2)^2+4\omega^2}\begin{bmatrix}10-\omega^2\\2\omega\end{bmatrix}$

$\boxed{y_p=\frac{1}{(10-\omega^2)^2+4\omega^2}\left[(10-\omega^2)\cos\omega t+2\omega\sin\omega t\right]}$

d) $A=\frac{\sqrt{(10-\omega^2)^2+4\omega^2}}{(10-\omega^2)^2+4\omega^2}=\frac{1}{\sqrt{(10-\omega^2)^2+4\omega^2}}$

$100-20\omega^2+\omega^4+4\omega^2$

$=(100-16\omega^2+\omega^4)^{-1/2}$ ✓

$A(0)=\frac{1}{\sqrt{10^2}}=\boxed{\frac{1}{10}}$

$A(3)=\frac{1}{\sqrt{(10-9)^2+4\cdot 9}}=\boxed{\frac{1}{\sqrt{37}}}$ ✓

≈ 1.644

e) $0=A'(\omega)=-\frac{1}{2}(100-16\omega^2+\omega^4)^{-3/2}\cdot(-32\omega+4\omega^3)$

$\rightarrow 0=\omega^3-8\omega=\omega(\omega^2-8)\rightarrow\omega=0,\sqrt{8}$

≈ 2.828 ✓

f)