MAT2705-01 08F Quiz 9 Print Name (Last, First) ____________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **You may use technology to find the roots of polynomials. [No mercy for mistakes here.]**

1. $x_1'(t) = -7\,x_1(t) + x_2(t)\,,\ x_2'(t) = 6\,x_1(t) - 6\,x_2(t),\ x_1(0) = 5,\ x_2(0) = -5$.

a) Rewrite this system of DEs and its initial conditions in matrix form for the vector variable $\vec{x} = \langle x_1, x_2 \rangle$, identifying the coefficient matrix A.

b) By hand showing all steps, find the standard eigenvectors $\vec{b}_1, \vec{b}_2$ produced by the solution algorithm with eigenvalues ordered in decreasing order. Evaluate the matrix $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$ and its inverse, and the diagonal matrix $A_D = B^{-1} A\, B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

c) Write out the new matrix DE for $\vec{y} = B^{-1}\vec{x}$, namely $\vec{y}\,' = A_D\,\vec{y}$, and write out and solve the three decoupled equations for the new scalar variables, then reconstruct the general solution $\vec{x} = B\,\vec{y}$.

d) Solve the initial conditions using matrix methods to find the IVP solution $\vec{x}(t)$.

e) Plot $x_1(t), x_2(t)$ on the same axes in an appropriate window and sketch what you see, labeling axes and tickmarks and curves. Find the value of t at which the two variables are equal. This can be done easily exactly using rules of exponents, but if you have trouble, find the numerical solution. What is this common value at that time?

f) **Optional.** [translate: read the answer key to see the picture].
On the grid provided, indicate by thick arrows both eigenvectors, extending them to labeled coordinate y_1, y_2 axes, and draw in the vector $\boldsymbol{x}(0)$ and its parallelogram projection onto those axes. Evaluate the new coordinates of $\boldsymbol{x}(0)$ using matrix methods. Do their values seem consistent with your drawing? Explain.

▶ solution

a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} -7 & 1 \\ 6 & -6 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

b) $0 = |A - \lambda I| = \begin{vmatrix} -7-\lambda & 1 \\ 6 & -6-\lambda \end{vmatrix} = (\lambda+7)(\lambda+6) - 6$

$= \lambda^2 + 13\lambda + 42 - 6 = \lambda^2 + 13\lambda + 36$

$= (\lambda+4)(\lambda+9) \rightarrow \lambda = -4, -9$

$\underline{\lambda = -4}$: $A + 4I = \begin{bmatrix} -7+4 & 1 \\ 6 & -6+4 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$

$\xrightarrow{\text{RREF}} \begin{array}{cc} L & F \end{array}$ $\begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 - \tfrac{1}{3} x_2 = 0 \rightarrow x_1 = t/3$, $x_2 = t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/3 \\ t \end{bmatrix} = t \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \quad \vec{b}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

$\underline{\lambda = -9}$: $A + 9I = \begin{bmatrix} -7+9 & 1 \\ 6 & -6+9 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \longrightarrow$ $\begin{array}{cc} L & F \end{array}$ $\begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + \tfrac{1}{2} x_2 = 0 \rightarrow x_1 = -\tfrac{1}{2} t$, $x_2 = t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

$\vec{b}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \quad B = \langle b_1 | b_2 \rangle = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix}$

b) continued: $B^{-1} = \dfrac{1}{\frac{1}{3} + \frac{1}{2}} \begin{bmatrix} 1 & 1/2 \\ -1 & 1/3 \end{bmatrix} = \dfrac{1}{5} \begin{bmatrix} 6 & 3 \\ -6 & 2 \end{bmatrix}$

$A_D = B^{-1} A B = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix}$ (Maple)

c) $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{matrix} y_1' = -4 y_1 & y_1 = c_1 e^{-4t} \\ y_2' = -9 y_2 & y_2 = c_2 e^{-9t} \end{matrix}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-4t} \\ c_2 e^{-9t} \end{bmatrix} = c_1 e^{-4t} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} + c_2 e^{-9t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}$$

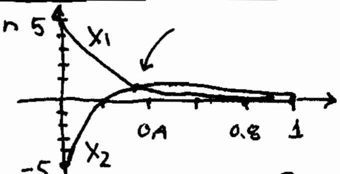
general solution.

d) $\begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix}}_{B} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 5 \\ -5 \end{bmatrix} \overset{\text{maple, mental}}{=} \begin{bmatrix} 3 \\ -8 \end{bmatrix}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 e^{-4t} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} - 8 e^{-9t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-4t} + 4e^{-9t} \\ 3e^{-4t} - 8e^{-9t} \end{bmatrix}}$$

ivp solution

e)

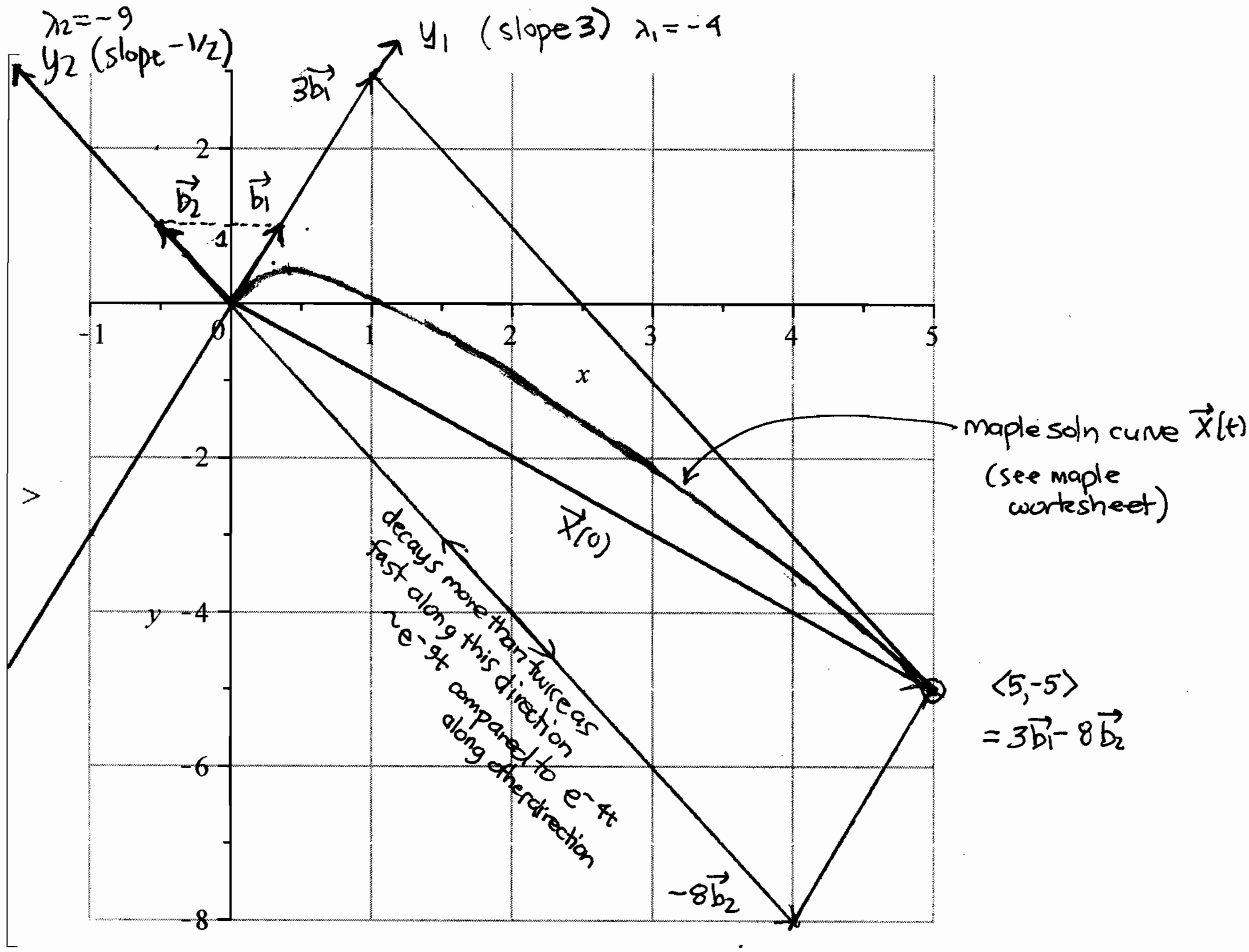


$e^{-4t} + 4e^{-9t} = 3e^{-4t} - 8e^{-9t} \rightarrow [12 e^{-9t} = 2e^{-4t}]\, e^{4t}/2$

$6 e^{-5t} = 1 \rightarrow e^{5t} = 6 \rightarrow \boxed{t = \tfrac{1}{5} \ln 6 \approx 0.358}$

$\hookrightarrow x_1 = e^{-\frac{4}{5}\ln 6} + 4 e^{-\frac{9}{5}\ln 6} = 6^{-4/5} + 4\,6^{-9/5}$

(maple) $= (5/18)\, 6^{1/5} \approx \boxed{0.397}$


$\lambda_2 = -9$
y_2 (slope $-1/2$)
y_1 (slope 3) $\lambda_1 = -4$
$3\vec{b_1}$
$\vec{b_2}$
$\vec{b_1}$
maple soln curve $\vec{X}(t)$
(see maple worksheet)
$\vec{X}(0)$
decays more than twice as fast along this direction $\sim e^{-9t}$ compared to e^{-4t} along other direction
$\langle 5,-5 \rangle$
$= 3\vec{b_1} - 8\vec{b_2}$
$-8\vec{b_2}$
x
y