MAT2705-01 ´08F Quiz 8 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **You may use technology to find the roots of polynomials. [No mercy for mistakes here.]**

1. $y''+4y'+8y=\cos(\omega t)$, $\omega \geq 0$.

a) Use the method of undetermined coefficients to find the steady state sinusoidal solution (the particular solution y_p) of this driven harmonic oscillator DE.

b) Evaluate the amplitude $A(\omega)$ of this sinusoidal function. Show that it simplifies to the formula

$$A(\omega) = \left(64+\omega^4\right)^{-\frac{1}{2}}.$$

c) Use calculus to show that the peak amplitude occurs at the value $\omega = 0$. Evaluate $A(0)$ exactly and numerically.

[This system has the value of the quality factor $Q = 1/\sqrt{2} \approx 0.707$, which is the lowest value at which there is a horizontal tangent on the amplitude plot for some nonnegative value of the frequency.]

d) Use technology to plot $A(\omega)$ for $\omega = 0..20$. Sketch what you see, labeling the axes, tickmarks and the vertical intercept.

e) Try to numerically solve for the value of the frequency ω for which $A(\omega) = \frac{1}{2}A(0)$, i.e., where the response amplitude is half the response at zero frequency.

▶ solution

a) y_h is associated with the complex roots $r=-2\pm 2i$, so no problem with root interference with RHS : $r=\pm i\omega$.

Trial y_p is general solution of $(D^2+\omega^2)y_p=0$:

$$\begin{aligned}
8\,[\,y_p &= c_3\cos\omega t + c_4 \sin\omega t\,]\\
4\,[\,y_p' &= -c_3\omega\sin\omega t + c_4\omega\cos\omega t\,]\\
1\,[\,y_p'' &= -c_3\omega^2\cos\omega t - c_4\omega^2\sin\omega t\,]
\end{aligned}$$

$$\begin{aligned}
y_p''+4y_p'+8y_p &= [(8-\omega^2)c_3 + 4\omega c_4]\cos\omega t &&= 1\cos\omega t\\
&\quad + [-4\omega c_3 + (8-\omega^2)c_4]\sin\omega t &&\quad + 0\sin\omega t
\end{aligned}$$

$$(8-\omega^2)c_3 + 4\omega c_4 = 1$$
$$-4\omega c_3 + (8-\omega^2)c_4 = 0$$

$$\begin{bmatrix} 8-\omega^2 & 4\omega \\ -4\omega & 8-\omega^2 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{\underbrace{(8-\omega^2)^2+16\omega^2}_{64-16\omega^2+\omega^4+16\omega^2 = 64+\omega^4}}\begin{bmatrix} 8-\omega^2 & -4\omega \\ 4\omega & 8-\omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{64+\omega^4}\begin{bmatrix} 8-\omega^2 \\ 4\omega \end{bmatrix}$$

$$\boxed{y_p = \frac{1}{64+\omega^2}\left[(8-\omega^2)\cos\omega t + 4\omega\sin\omega t\right]}$$

b) $A(\omega) = \sqrt{c_3^2+c_4^2} = \dfrac{1}{64+\omega^2}\sqrt{(8-\omega^2)^2+16\omega^2} = \dfrac{\sqrt{64+\omega^4}}{64+\omega^4} = \boxed{\dfrac{1}{\sqrt{64+\omega^4}}}$

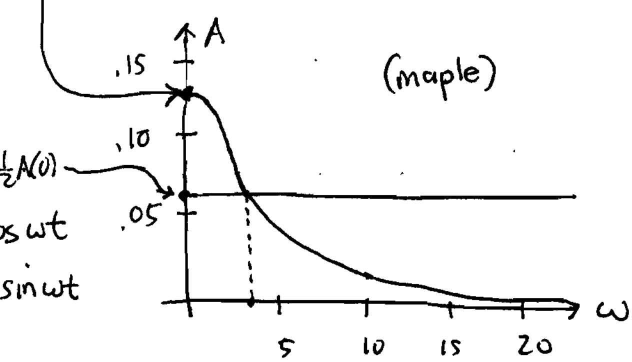
c) $A(\omega) = (64+\omega^4)^{-1/2}$

$A'(\omega) = -\frac{1}{2}(64+\omega^4)^{-3/2}(0+4\omega^3)$

$= -\dfrac{2\omega^3}{(64+\omega^4)^{3/2}} = 0 \rightarrow \boxed{\omega = 0}$

$\boxed{A(0) = 64^{-1/2} = 1/8} \boxed{= 0.125}$

d)



e) $A(\omega) = \frac{1}{2}A(0)$: $(64+\omega^4)^{-1/2} = \frac{1}{16}$

numerical solve : $\omega \approx 3.72$ (maple)

(exactly: $64+\omega^4 = 16^2$

$\omega^4 = 16^2 - 64 = 3\cdot 64 = 12\cdot 16$

$\omega = 2\cdot 12^{\frac{1}{4}}$)