MAT 2705-01 08F Final Exam Answers (1)

① a) $\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\vec{x}\,'} = \begin{bmatrix} -4x_1+8x_2 \\ x_1-6x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 & 8 \\ 1 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

b) $|A-\lambda I| = \begin{vmatrix} -4-\lambda & 8 \\ 1 & -6-\lambda \end{vmatrix} = (\lambda+4)(\lambda+6)-8$

$= \lambda^2+10\lambda+24-8 = \lambda^2+10\lambda+16$

$= (\lambda+2)(\lambda+8) = 0 \longrightarrow \lambda = -2, -8$

$\lambda = -2$: $A+2I = \begin{bmatrix} -4+2 & 8 \\ 1 & -6+2 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 1 & -4 \end{bmatrix}$

RREF $\rightarrow$ $\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (L F) $\quad x_1 = 4x_2 = 4t$, $\quad x_2 = t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix} = t\,\vec{b}_1$

$\lambda = -8$: $A+8I = \begin{bmatrix} -4+8 & 8 \\ 1 & -6+8 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$

RREF $\rightarrow$ $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (L F) $\quad x_1 = -2x_2 = -2t$, $\quad x_2 = t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix} = t\,\vec{b}_2$

$\lambda = -2, -8 = \lambda_1, \lambda_2$

$B = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = \langle \vec{b}_1 | \vec{b}_2 \rangle$

$B^{-1} = \frac{1}{4+2}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

Note $B^{-1}AB = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -8 \end{bmatrix} \equiv A_D$

c) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \propto \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\quad \frac{x_2}{x_1} = \frac{1}{4} = \text{slope}$ $\quad \boxed{x_2 = \frac{1}{4}x_1}$ (λ_1)

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \propto \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\quad \frac{x_2}{x_1} = \frac{1}{-2} = \text{slope}$ $\quad \boxed{x_2 = -\frac{1}{2}x_1}$ (λ_2)

d) $\vec{x} = B\vec{y}, \quad \vec{y} = B^{-1}\vec{x}$

$\vec{y}(0) = B^{-1}\vec{x}(0) = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}}$

(see graph)

e) $\vec{x}\,' = Ax \rightarrow B^{-1}[(B\vec{y})' = AB\vec{y}] \rightarrow \vec{y}\,' = \underbrace{(B^{-1}AB)}_{A_D}\vec{y}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2y_1 \\ -8y_2 \end{bmatrix}$ $\quad y_1' = -2y_1 \quad y_1 = c_1e^{-2t}$, $\quad y_2' = -8y_2 \quad y_2 = c_2e^{-8t}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{-2t} \\ c_2e^{-8t} \end{bmatrix} = \begin{bmatrix} 4c_1e^{-2t} - 2c_2e^{-8t} \\ c_1e^{-2t} + c_2e^{-8t} \end{bmatrix}$ gen soln.

f) $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = B\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = B\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ see above.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4e^{-2t} - 4e^{-8t} \\ e^{-2t} + 2e^{-8t} \end{bmatrix}$ IVP soln

$= e^{-2t}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 2e^{-8t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (eigenvector form)

$x_1 = 4e^{-2t} - 4e^{-8t}, \quad x_2 = e^{-2t} + 2e^{-8t}$ (scalar form)

g) $k = 2, 8 \rightarrow \tau = \frac{1}{2}, \frac{1}{8} = \tau_1, \tau_2$ $\quad \frac{1}{2} > \frac{1}{8}$ is longer.

$5\tau_1 = 5/2 = 2.5$ $\quad$ (see plot).

② a) $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_1+8x_2 \\ x_1-6x_2 \end{bmatrix}$

$x_1'' = -4x_1+8x_2, \quad x_2'' = x_1-6x_2, \quad x_1(0) = 0, \quad x_1'(0) = 0$
$x_2(0) = 3, \quad x_2'(0) = 0$

b) $\vec{x}\,'' = A\vec{x} \xrightarrow[\text{as above}]{\text{same}} \vec{y}\,'' = A_D\vec{y}$

$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2y_1 \\ -8y_2 \end{bmatrix}$ $\quad y_1'' = -2y_1, \quad y_2'' = -8y_2$

Note: $\sqrt{8} = \sqrt{2\cdot 4} = 2\sqrt{2}$

$y_1'' + 2y_1 = 0 \quad y_1 = c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t$
$y_2'' + 8y_2 = 0 \quad y_2 = c_3\cos 2\sqrt{2}t + c_4\sin 2\sqrt{2}t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t \\ c_3\cos 2\sqrt{2}t + c_4\sin 2\sqrt{2}t \end{bmatrix}$

$= (c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t)\begin{bmatrix} 4 \\ 1 \end{bmatrix} + (c_3\cos 2\sqrt{2}t + c_4\sin 2\sqrt{2}t)\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

c)

$\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}}_{B}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = B^{-1}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (see above) $\quad t=0$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\sqrt{2}c_1\sin\sqrt{2}t + \sqrt{2}c_2\cos\sqrt{2}t \\ -2\sqrt{2}c_3\sin 2\sqrt{2}t + 2\sqrt{2}c_4\cos 2\sqrt{2}t \end{bmatrix}$

$= \underbrace{\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}}_{B\text{ invertible}}\begin{bmatrix} \sqrt{2}c_2 \\ 2\sqrt{2}c_4 \end{bmatrix} \rightarrow \begin{bmatrix} \sqrt{2}c_2 \\ 2\sqrt{2}c_4 \end{bmatrix} = B^{-1}\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cos\sqrt{2}t\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 2\cos 2\sqrt{2}t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 4\cos\sqrt{2}t - 4\cos 2\sqrt{2}t \\ \cos\sqrt{2}t + 2\cos 2\sqrt{2}t \end{bmatrix}$

$x_1 = 4\cos\sqrt{2}t - 4\cos 2\sqrt{2}t$
$x_2 = \cos\sqrt{2}t + 2\cos 2\sqrt{2}t.$

MAT 2705-01 08F Final Exam Answers (2)

② d) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\cos\sqrt{2}\,t}_{\omega_{tandem}} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \underbrace{2\cos 2\sqrt{2}\,t}_{\omega_{accordian}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

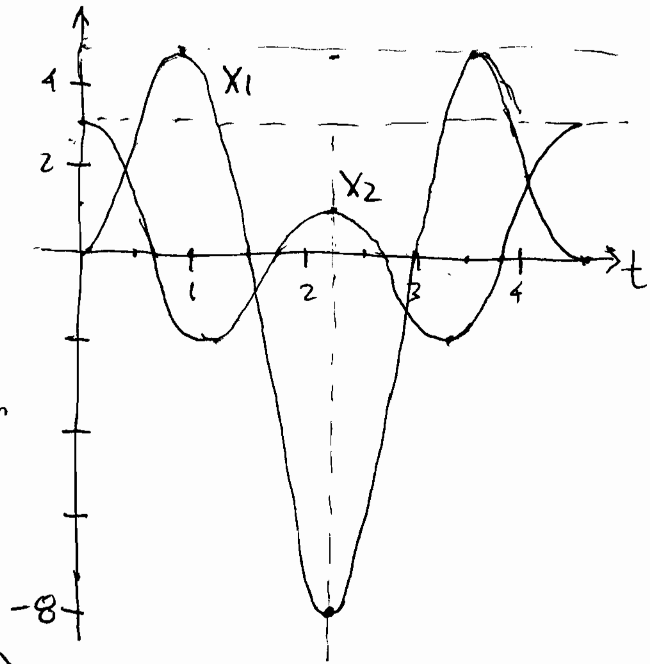
$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$: same sign: tandem

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$: opp sign; accordian,

$$T_{tandem} = \frac{2\pi}{\omega_{tandem}} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\,\pi \approx 4.443$$

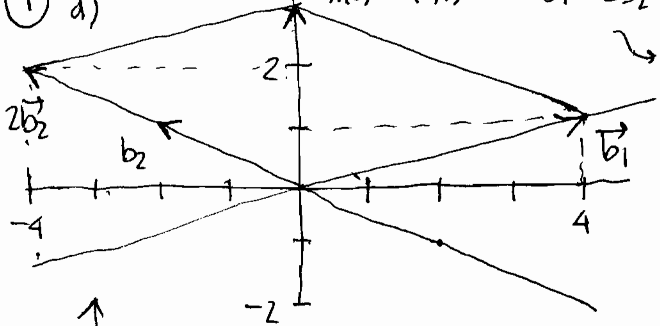
$$T_{accordian} = \frac{2\pi}{\omega_{accordian}} = \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}} = \frac{\sqrt{2}}{2}\pi = \frac{1}{2} T_{tandem}$$

$\underbrace{T_{acc} \quad T_{acc}}_{T_{tandem}}$ } 2 accordian periods fit into the tandem period which is the common period.



① d) $\vec{x}(0) = \langle 0,3 \rangle = \vec{b}_1 + 2\vec{b}_2$ consistent.

$\vec{y}(0) = \langle 1,2 \rangle$



g)