# MAT2705-01 08F TEST 1 ANSWERS

① a) $\frac{dy}{dx} = 3xy^{1/2}$ (separable)

$\int \frac{dy}{y^{1/2}} = \int 3x\,dx$ (separate)

$\int y^{-1/2}dy = 3\frac{x^2}{2} + C_1$ (integrate)

$\frac{y^{1/2}}{1/2} = \frac{3}{2}x^2 + C_1$

$y^{1/2} = \frac{1}{2}\left(\frac{3}{2}x^2 + C_1\right)$ (solve for y, rename $\frac{1}{2}C_1 = C$)

$= \frac{3}{4}x^2 + C$

$\boxed{y = \left(\frac{3}{4}x^2 + C\right)^2}$ gen soln

$= \frac{9}{16}x^4 + \frac{3}{2}x^2C + C^2$

squaring step introduced extra solns not allowed: $C_1 = 0 + C \checkmark$

b) $9 = y(0) = C^2 \rightarrow C = 3$

$3 = y(0)^{1/2} = C \rightarrow$ positive soln.

$\boxed{y = \left(\frac{3}{4}x^2 + 3\right)^2}$ IVP soln

$= \frac{9}{16}(x^2+4)^2$

c) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3}{4}x^2+3\right)^2 = 2\left(\frac{3}{4}x^2+3\right)^1\left(\frac{3}{4}(2x)\right)$

$= 3x\left(\frac{3}{4}x^2+3\right)$

$\frac{dy}{dx} = 3xy^{1/2}$

$3x\left(\frac{3}{4}x^2+3\right) = 3x\left(\frac{3}{4}x^2+3\right)$ ✓

d) $36 = \left(\frac{3}{4}x^2+3\right)^2$

$6 = \frac{3}{4}x^2 + 3$

$3 = \frac{3}{4}x^2$

$4 = x^2 \rightarrow x = \pm 2 \rightarrow 2$ since want $x > 0$

$\boxed{x = 2}$

→ ① e) The graphical soln is dead on, going right through the predicted point.

② a) $\left[\frac{dx}{dt} + \frac{x}{16} = \frac{1}{4}\right] \rightarrow \frac{d}{dt}\left(xe^{\frac{t}{16}}\right) = \frac{1}{4}e^{\frac{t}{16}}$

$\int \frac{1}{16}dt = \frac{t}{16}$

$e^{t/16}$ integrating factor

$\left[xe^{t/16} = \frac{1}{4}\frac{e^{t/16}}{1/16} + C\right]e^{-t/16}$

$\boxed{x = 4 + Ce^{-t/16}}$ general soln

$20 = x(0) = 4 + C \rightarrow C = 16$

$\boxed{x = 4 + 16e^{-t/16}}$ IVP soln

② b) $\lim_{t\to\infty} x = \lim_{t\to\infty} 4 + 16\underbrace{e^{-t/16}}_{\to 0} = 4$

x approaches 4 asymptotically (from above)

c) $\frac{x(t)}{8000} = .0010$ → $4 + 16e^{-\frac{t}{16}} = (.001)(8000) = 8$

$\frac{4+16e^{-t/16}}{8000}$

$16e^{-t/16} = 4$

$e^{-t/16} = 1/4$

$-\frac{t}{16} = \ln 1/4 = \ln 4^{-1} = -\ln 4$, $t = -16(-\ln 4)$

$= 16\ln 4 = 16\ln 2^2$

$= 32\ln 2$

$\approx 22.1807$

$\approx$ **22.2 days**

"It takes about 22.2 days to reduce the pollutant concentration from 0.25% to 0.10%."

e) $e^{-\frac{t}{16}} \leftrightarrow \boxed{\tau = 16 \text{ days}}$ "characteristic time"

$e^{-\frac{t}{16}} = \frac{1}{2} \rightarrow -\frac{t}{16} = \ln 2^{-1} = -\ln 2 \rightarrow t = \boxed{16\ln 2 = T_{1/2} \approx 11.090}$ "half-life"

part c) $t$ = 2 half-lives, (0.25% reduces by $1/4 = 1/2^2$)