MAT2705-01/04 07S Quiz 8 Print Name (Last, First) ______________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. IVP: $\vec{x}\,'' = A\,\vec{x} + \vec{f}$, $\vec{x}(0) = 0$, $\vec{x}\,'(0) = 0$.

For the 2 mass 3 spring problem let $\vec{f} = \langle 2\cos(t), 0\rangle$ and set $m_1 = m_2 = 1$, $k_1 = 4 = k_3$, $k_2 = 6$ in

$$A = \begin{bmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} \end{bmatrix}.$$

The eigenvalues of A are $\lambda = -4, -16$ and the corresponding matrix of eigenvectors is $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. a) Introducing the coordinate change $\vec{x} = B\,\vec{y}$, derive or remember (correctly) the DE for $\vec{y}$. b) Write out and solve the decoupled equations for y_1 and y_2 and then evaluate $\vec{x}$. c) Impose the initial conditions on your general solution and express your final answer first in the eigenvector form $\vec{x} = y_1\,\vec{b}_1 + y_2\,\vec{b}_2$ which shows the two modes of the system and then in the scalar form $x_1 = \ldots,\ x_2 = \ldots$ where all like terms have been combined.

► solution

1 a) $A = \begin{bmatrix} -\frac{4+6}{1} & \frac{6}{1} \\ \frac{6}{1} & -\frac{6+4}{1} \end{bmatrix} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}$

$\left(\text{so } \begin{array}{l} x_1'' = -10x_1 + 6x_2 + 2\cos t \\ x_2'' = 6x_1 - 10x_2 \end{array}\right)$

$A_B = B^{-1}AB = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix}$

$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $B^{-1}\vec{f} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2\cos t \\ 0 \end{bmatrix} = \begin{bmatrix} \cos t \\ -\cos t \end{bmatrix}$

$\cdot\ \vec{x}\,'' = A\vec{x} + \vec{f} \rightarrow B^{-1}[(B\vec{y})'' = AB\vec{y} + \vec{f}\,]$

$\vec{x} = B\vec{y}$

$\vec{y}\,'' = B^{-1}AB\,\vec{y} + B^{-1}\vec{f}$

$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \cos t \\ -\cos t \end{bmatrix}$

$y_1'' = -4y_1 + \cos t$

$y_2'' = -16y_2 - \cos t$

$\boxed{\begin{array}{l} y_1'' + 4y_1 = \cos t \\ y_2'' + 16y_2 = -\cos t \end{array}}$ $\begin{array}{l} \rightarrow r^2 + 4 = 0,\ r = \pm 2i \\ \rightarrow r^2 + 16 = 0,\ r = \pm 4i \end{array}$

$y_{1h} = c_1\cos 2t + c_2\sin 2t$ $\qquad y_{1p} = c_5\cos t$

$y_{2h} = c_3\cos 4t + c_4\sin 4t$ $\qquad y_{2p} = c_6\cos t$

$y_{1p}'' + 4y_{1p} = (-c_5 + 4c_5)\cos t = 3c_5\cos t = \cos t$

$y_{2p}'' + 16y_{2p} = (-c_6 + 16c_6)\cos t = 15c_6\cos t = -\cos t$

$3c_5 = 1,\ 15c_6 = -1 \rightarrow c_5 = \frac{1}{3},\ c_6 = -\frac{1}{15}$

$\boxed{\begin{array}{l} y_1 = c_1\cos 2t + c_2\sin 2t + \frac{1}{3}\cos t \\ y_2 = c_3\cos 4t + c_4\sin 4t - \frac{1}{15}\cos t \end{array}} \rightarrow \boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (c_1\cos 2t + c_2\sin 2t + \frac{1}{3}\cos t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3\cos 4t + c_4\sin 4t - \frac{1}{15}\cos t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$\vec{x}\,' = (-2c_1\sin 2t + 2c_2\cos 2t - \frac{1}{3}\sin t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-4c_3\sin 4t + 4c_4\cos 4t + \frac{1}{15}\sin t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

c) $\vec{x}(0) = (c_1 + \frac{1}{3})\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3 - \frac{1}{15})\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 - c_3 + \frac{1}{3} + \frac{1}{15} \\ c_1 + c_3 + \frac{1}{3} - \frac{1}{15} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} -6/15 \\ -4/15 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\left(-\frac{2}{15}\right)\begin{bmatrix} 3 \\ 2 \end{bmatrix} = -\frac{1}{15}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/15 \end{bmatrix}$

$\vec{x}\,'(0) = 2c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4c_4\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2c_2 \\ 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2c_2 \\ 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow c_2 = 0 = c_4$.

$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-\frac{1}{3}\cos 2t + \frac{1}{3}\cos t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (\frac{1}{15}\cos 4t - \frac{1}{15}\cos t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \begin{bmatrix} -\frac{1}{3}\cos 2t - \frac{1}{15}\cos 4t + (\frac{1}{3} + \frac{1}{15})\cos t \\ -\frac{1}{3}\cos 2t + \frac{1}{15}\cos 4t + (\frac{1}{3} - \frac{1}{15})\cos t \end{bmatrix}$

$\frac{2}{15}$ $\quad$ $\frac{4}{15}$

$\boxed{x_1 = -\frac{1}{3}\cos 2t - \frac{1}{15}\cos 4t + \frac{2}{5}\cos t\ ,\quad x_2 = -\frac{1}{3}\cos 2t + \frac{1}{15}\cos 4t + \frac{4}{15}\cos t}$