MAT2705-01/04 07S Quiz 7 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **You may use technology to find the integer roots of the characteristic equation. [No mercy for mistakes here.]**

1. $2y'' + 12y' + 180y = 0, y(0) = 1, y'(0) = -15$.

a) Identify the values of the characteristic decay time τ_0, natural frequency ω_0, and their product, the quality factor Q for this damped harmonic oscillator system.

b) Find the general solution $y(t)$ by hand and identify its characteristic decay time τ, quasifrequency ω and quasiperiod $T = 2\pi/\omega$.

c) Find the solution which satisfies the initial conditions, showing all work.

d) Rewrite your sinusoidal factor of the solution (everything but the exponential factor) of the IVP in phase-shifted cosine form $A\cos(\omega t - \delta)$ and state the initial amplitude A and phase shift δ.

e) Use technology to plot your result of part c) for $t = 0..5\,\tau$ [not from part d)!], including the envelope functions $\pm A\,e^{-kt}$, and make a rough sketch of what you see, labeling the axes with variable names and key tickmarks on your sketch. Does the envelope you found fit the solution curve as it should? [Yes or no, with explanation would be a good response.]

Optional. Think about ***only if*** you finish early: Evaluate the ratio of 5 times the decay time τ to the quasiperiod T of your exponentially decaying sinusoidal solution function in order to get the number of quasiperiods that fit into 5 characteristic times. Does this agree with your plot?

▶ solution

a) $\frac{1}{2}[2y'' + 12y' + 180y = 0] \rightarrow y'' + \overset{k_0}{6}y' + \overset{\omega_0^2}{90}y = 0$ (standard form) $\rightarrow \tau_0 = \frac{1}{k_0} = \frac{1}{6} \approx .167$

$\omega_0 = \sqrt{90} = 3\sqrt{10} \approx 9.49$, $Q = \omega_0\tau_0 = \frac{3\sqrt{10}}{6} = \frac{\sqrt{10}}{2} \approx 1.58$

$y = e^{rt}$

b) $r^2 + 6r + 90 = 0$, $r = \frac{-6 \pm \sqrt{36 - 4\cdot 90}}{2} = -3 \pm \frac{\sqrt{36(1-10)}}{2} = -3 \pm \frac{6}{2}\sqrt{-9} = -3 \pm 9i$

$\boxed{y = e^{-3t}(c_1 \cos 9t + c_2 \sin 9t)}$

$\boxed{\omega = 9}$ $k = 3 \rightarrow \boxed{\tau = \frac{1}{k} = \frac{1}{3}}$

$\boxed{T = \frac{2\pi}{9} \approx 0.698}$

c) $y' = -3e^{-3t}(c_1\cos 9t + c_2 \sin 9t) + e^{-3t}(-9c_1 \sin 9t + 9c_2 \cos 9t)$

$y(0) = c_1 = 1$

$y'(0) = -3c_1 + 9c_2 = -15 \rightarrow c_2 = \frac{-15 + 3(1)}{9} = -\frac{12}{9} = -\frac{4}{3}$

$\boxed{y = e^{-3t}(\cos 9t - \frac{4}{3}\sin 9t)}$

d) $A = \sqrt{1 + (\frac{4}{3})^2} = \sqrt{1 + \frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$

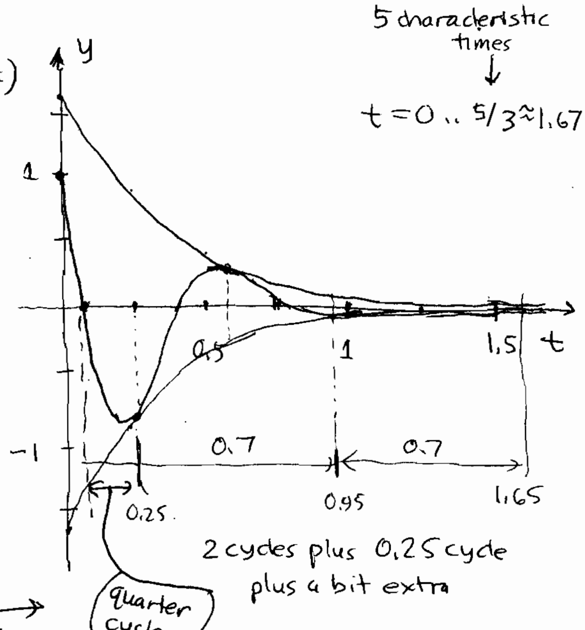
$\delta = -\arctan(4/3) \approx -53.1°$

$(1, -4/3)$

$\boxed{y = \frac{5}{3}e^{-3t}\cos(9t + \arctan 4/3)}$

5 characteristic times

$t = 0 .. 5/3 \approx 1.67$



2 cycles plus 0.25 cycle plus a bit extra

optional $\frac{5\tau}{T} = \frac{5/3}{2\pi/9} \approx 2.39$ cycles per 5 characteristic times $\rightarrow$ consistent with what we see