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① a) $x'' + 6x' + 8x = 40\cos 2t$

$x_h = e^{rt} \nearrow (r^2+6r+8)e^{rt} = 0$

$(r+2)(r+4) = 0 \to r = -2, -4$

$e^{rt} = e^{-2t}, e^{-4t}$

$x_h = c_1 e^{-2t} + c_2 e^{-4t}$ (overdamped)

$8\,[x_p = c_3\cos 2t + c_4\sin 2t]$

$6\,[x_p' = -2c_3\sin 2t + 2c_4\cos 2t]$

$1\,[x_p'' = -4c_3\cos 2t - 4c_4\sin 2t]$

$x_p'' + 6x_p' + 8x_p = [(8-4)c_3 + 12c_4]\cos 2t + [-12c_3 + [8-4]c_4]\sin 2t$

$= (4c_3 + 12c_4)\cos 2t + (-12c_3 + 4c_4)\sin 2t = 40\cos 2t$

$4c_3 + 12c_4 = 40$

$-12c_3 + 4c_4 = 0$

$\begin{bmatrix}4 & 12\\ -12 & 4\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}40\\ 0\end{bmatrix} \quad 4\begin{bmatrix}1 & 3\\ -3 & 1\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}40\\ 0\end{bmatrix}$

$\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \frac{1}{4}\,\frac{1}{10}\begin{bmatrix}1 & -3\\ 3 & 1\end{bmatrix}\begin{bmatrix}40\\ 0\end{bmatrix} = \frac{1}{4}\,\frac{1}{10}\begin{bmatrix}1\cdot 40\\ 3\cdot 40\end{bmatrix} = \begin{bmatrix}1\\ 3\end{bmatrix}$

$x = x_h + x_p$

$\boxed{x = c_1e^{-2t} + c_2e^{-4t} + \cos 2t + 3\sin 2t}$ gen soln

b) $x' = -2c_1e^{-2t} - 4c_2e^{-4t} - 2\sin 2t + 6\cos 2t$

$x(0) = c_1 + c_2 + 1 = 0$

$x'(0) = -2c_1 - 4c_2 + 6 = 10$

$\begin{bmatrix}1 & 1\\ -2 & -4\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}-1\\ 4\end{bmatrix}$

$\begin{bmatrix}c_1\\ c_2\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}-4 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}-1\\ 4\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0\\ 2\end{bmatrix} = \begin{bmatrix}0\\ -1\end{bmatrix}$

$\boxed{x = -e^{-4t} + \cos 2t + 3\sin 2t}$

IVP solution

c) $\tau = \frac{1}{4} \to 4.6\left(\frac{1}{4}\right) \approx 1.15$

$T = \frac{2\pi}{2} = \pi \approx 3.14 \qquad \frac{1.15}{3.14} \approx 0.367$

roughly 1/3 period, transient decays to 1% of initial value

② a) $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}' = \underbrace{\begin{bmatrix}-9 & -4 & 0\\ 6 & 1 & 0\\ -6 & -4 & -3\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \quad \begin{bmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$

$\vec{x}' = A\vec{x}$

b) $|A - \lambda I| = \begin{vmatrix}-9-\lambda & -4 & 0\\ 6 & 1-\lambda & 0\\ -6 & -4 & -3-\lambda\end{vmatrix} \overset{\text{maple}}{=} -\lambda^3 - 11\lambda^2 - 39\lambda - 45$

$= -(\lambda+5)(\lambda+3)^2 = 0$

$\lambda = -5, -3, -3$

2b) $\lambda = -5 \quad A + 5I = \begin{bmatrix}-9+5 & -4 & 0\\ 6 & 1+5 & 0\\ -6 & -4 & -3+5\end{bmatrix} = \begin{bmatrix}-4 & -4 & 0\\ 6 & 6 & 0\\ -6 & -4 & 2\end{bmatrix} \to$

$\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ -6 & -4 & 2\end{bmatrix} \to \begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ -2 & 0 & 2\end{bmatrix} \to \begin{bmatrix}1 & 1 & 0\\ 1 & 0 & -1\\ 0 & 0 & 0\end{bmatrix} \to \begin{array}{ccc}L & L & F\end{array}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$x_1 - x_3 = 0 \to x_1 = t$

$x_2 + x_3 = 0 \to x_2 = -t$

$x_3 = t$

$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}t\\ -t\\ t\end{bmatrix} = t\underbrace{\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}}_{\vec{b}_1}$

$\lambda = -3 \quad A + 3I = \begin{bmatrix}-9+3 & -4 & 0\\ 6 & 1+3 & 0\\ -6 & -4 & -3+3\end{bmatrix} = \begin{bmatrix}-6 & -4 & 0\\ 6 & 4 & 0\\ -6 & -4 & 0\end{bmatrix}$

$\to \begin{array}{ccc}L & F & F\end{array}\begin{bmatrix}1 & 2/3 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

$x_1 + \frac{2}{3}x_2 = 0 \to x_1 = -\frac{2}{3}t_1$

$x_2 = t_1$

$x_3 = t_2$

$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}t_1\\ t_1\\ t_2\end{bmatrix} = t_1\underbrace{\begin{bmatrix}-2/3\\ 1\\ 0\end{bmatrix}}_{\vec{b}_2} + t_2\underbrace{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}}_{\vec{b}_3}$

$\boxed{\lambda = -5, -3, -3 \quad B = \begin{bmatrix}1 & -2/3 & 0\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}} \to A_B = B^{-1}AB = \begin{bmatrix}-5 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{bmatrix}$

$x = By, \quad y = B^{-1}x$

$x' = Ax \to B^{-1}(By)' = ABy \to y' = (B^{-1}AB)y = A_By$

$\begin{bmatrix}y_1'\\ y_2'\\ y_3'\end{bmatrix} = \begin{bmatrix}-5 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$

$y_1' = -5y_1 \qquad y_1 = c_1e^{-5t}$

$y_2' = -3y_2 \qquad y_2 = c_2e^{-3t}$

$y_3' = -3y_3 \qquad y_3 = c_3e^{-3t}$

$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1 & -2/3 & 0\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}c_1e^{-5t}\\ c_2e^{-3t}\\ c_3e^{-3t}\end{bmatrix} = \boxed{\begin{bmatrix}c_1e^{-5t} - \frac{2}{3}c_2e^{-3t}\\ -c_1e^{-5t} + c_2e^{-3t}\\ c_1e^{-5t} + c_3e^{-3t}\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}}$

general soln

c) $\begin{bmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix} = \begin{bmatrix}1 & -2/3 & 0\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix} \qquad \begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix} = \begin{bmatrix}1 & -2/3 & 0\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$

$\overset{\text{maple}}{=} \begin{bmatrix}3 & 2 & 0\\ 3 & 3 & 0\\ -3 & -2 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix} = \begin{bmatrix}5\\ 6\\ -5\end{bmatrix} \qquad \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}5e^{-5t} - (2/3)6\,e^{-3t}\\ -5e^{-5t} + 6e^{-3t}\\ +5e^{-5t} - 5e^{-3t}\end{bmatrix} = \cancel{\begin{bmatrix}5e^{-5t} - 4e^{-3t}\\ -5e^{-5t} + 6e^{-3t}\\ 5e^{-5t} - 5e^{-3t}\end{bmatrix}}$

IVP solution $\boxed{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}5e^{-5t} - 4e^{-3t}\\ -5e^{-5t} + 6e^{-3t}\\ 5e^{-5t} - 5e^{-3t}\end{bmatrix}}$ more legible

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③ a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -7 & 4 \\ 3 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$, $\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b) $|A-\lambda I| = \begin{vmatrix} -7-\lambda & 4 \\ 3 & -3-\lambda \end{vmatrix} = (\lambda+7)(\lambda+3) - 12 = \lambda^2 + 10\lambda + \underbrace{21-12}_{=9} = 0 = (\lambda+9)(\lambda+1) = 0$

$\lambda = -9, -1$:

$\lambda = -9$ $\quad A+9I = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \rightarrow \begin{array}{c} L\ F \\ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$, $x_2 = t \rightarrow x_1 = -2t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda = -1$: $A+I = \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \rightarrow \begin{array}{c} L\ F \\ \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - \frac{2}{3}x_2 = 0$, $x_2 = t \rightarrow x_1 = \frac{2}{3}t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

$\lambda = -9, -1$

$B = \begin{bmatrix} -2 & 2/3 \\ 1 & 1 \end{bmatrix}$ $\quad A_B = B^{-1}AB = \begin{bmatrix} -9 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad B^{-1} = -\frac{3}{8}\begin{bmatrix} 1 & -2/3 \\ -1 & -2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3 & 2 \\ 3 & 6 \end{bmatrix}$

$\vec{x}'' = A\vec{x} \longrightarrow \vec{y}'' = A_B\vec{y}$

$\vec{x} = B\vec{y}$

$\vec{y} = B^{-1}\vec{x}$

$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad y_1'' = -9y_1 \quad y_1'' + 9y_1 = 0 \longrightarrow y_1 = c_1\cos 3t + c_2 \sin 3t$

$\quad y_2'' = -1y_2 \quad y_2'' + y_2 = 0 \quad\quad y_2 = c_3 \cos t + c_4 \sin t$

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (c_1\cos 3t + c_2 \sin 3t)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + (c_3\cos t + c_4 \sin t)\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1\cos 3t - 2c_2\sin 3t + \frac{2}{3}c_3\cos t + \frac{2}{3}c_4 \sin t \\ c_1\cos 3t + c_2\sin 3t + c_3\cos t + c_4\sin t \end{bmatrix} \text{ gen soln}}$$

c) $\vec{x}' = (-3c_1\sin 3t + 3c_2\cos 3t)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + (-c_3\sin t + c_4 \cos t)\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

$\vec{x}(0) = c_1\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 2/3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$ $\quad \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 8 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$

$\vec{x}'(0) = 3c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_4\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 2/3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad \begin{bmatrix} 3c_2 \\ c_4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \rightarrow$

$\rightarrow c_2 = 0$, $c_4 = 3$

$\vec{x} = \underbrace{-\cos 3t}_{A_1 = 1}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \underbrace{(9\cos t + 3\sin t)}_{A_2 = 3\sqrt{1+3^2} = 3\sqrt{10} \approx 9.49}\begin{bmatrix} 2/3 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} +2\cos 3t + 6\cos t + 2\sin t \\ -\cos 3t + 9\cos t + 3\sin t \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

IVP solution

$\delta_1 = \pi$ $\quad -1$

$3 \quad 9$ $\quad \delta_2 = \arctan 1/3 \approx 0.322$

d) y_1 is the accordian mode since $\vec{b}_1$ has opposite signed entries

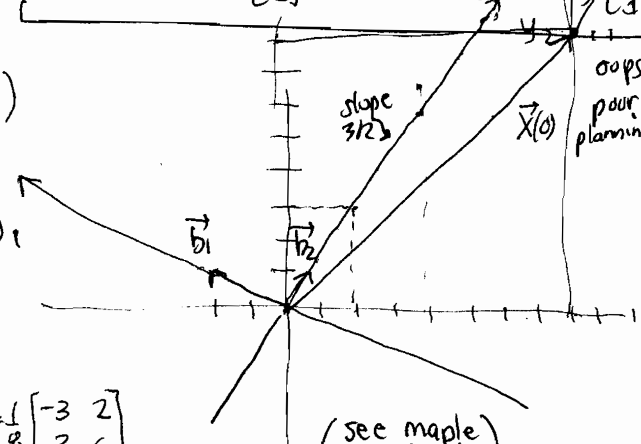
$\omega_1 = 3$, $T_1 = \frac{2\pi}{3}$.

y_2 is the tandem mode since $\vec{b}_2$ has same sign entries.

$\omega_2 = 1$, $T_2 = 2\pi$.

e) $$\boxed{\vec{x} = -\cos 3t\begin{bmatrix} -2 \\ 1 \end{bmatrix} + (9\cos t + 3\sin t)\begin{bmatrix} 2/3 \\ 4 \end{bmatrix} = \cos(3t-\pi)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\sqrt{10}\cos(t - \arctan 1/3)\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}}$$

f)



(see maple worksheet.)