MAT2705-01/04 07S Take Home Test 3 Print Name (Last, First) ________________________________ |__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions and root finding.** Print requested technology plots, annotate them appropriately and attach to the relevant problems.

1. The displacement $y(t)$ of a damped harmonic oscillator system satisfies

$$m\,y''(t) + c\,y'(t) + K\,y(t) = F(t).$$

Let $m = 1$, $c = 10$, $K = 650$ in MKS units. Consider the initial conditions $y(0) = 0$, $y'(0) = 0$ and the following driving force functions $F(t)$:

a) $F(t) = 100$.

Find the initial value problem solution by hand and evaluate the asymptotic value $y_\infty = \lim_{t \to \infty} y(t)$. Make a single plot in an appropriate viewing window showing both the solution function and its horizontal asymptote.

b) $F(t) = 100\, e^{-\frac{t}{5}}$.

Find the initial value problem solution by hand and evaluate the asymptotic value $y_\infty = \lim_{t \to \infty} y(t)$. Make a two plots in appropriate viewing windows showing both the solution function and its horizontal asymptote: first to show clearly the initial oscillating behavior, and second to show the approach to the asymptotic value.

c) $F(t) = 100 \cos(25\, t)$.

Find the initial value problem solution by hand. Make a single plot in an appropriate viewing window showing both the solution function and the steady state solution. Evaluate the values of the amplitude and phase shift for the steady state solution (the part of the solution which remains after the transient has died away) and express the phase shift in radians, degrees and cycles.

d) What are the natural frequency ω_0, natural decay time τ_0, and the quality factor $Q = \omega_0 \tau_0$ (both exact and numeric values) for this system?

e) $F(t) = 100 \cos(\omega\, t)$.

Explore resonance for this system by finding the steady state solution by hand, where the nonegative frequency ω of the driving force function is a parameter.

f) Evaluate the steady state amplitude function $A(\omega)$ and use calculus to find the exact and numerical value of the frequency ω_p and the amplitude $A(\omega_p)$ where it has its peak value for $\omega \geq 0$.

What is the numerical value of the ratio $A(\omega_p)/A(0)$? How does this compare to Q? Does the value of your steady state amplitude for part c) agree with $A(25)$?

g) Plot this amplitude function $A(\omega)$ in an appropriate window (showing the behavior of the entire function for $\omega \geq 0$) together with the constant functions $A(\omega_0)$, $A(\omega_p)$ and $A(0)$ and hand annotate on your axes the values of these frequencies and amplitudes.

2. $x_1'(t) = -5\, x_1(t) + 6\, x_2(t) + 6\, x_3(t)$, $x_2'(t) = x_1(t) - 4\, x_2(t) - 2\, x_3(t)$, $x_3'(t) = -3\, x_1(t) + 6\, x_2(t) + 4\, x_3(t)$, $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 1$.

a) Write this system and its initial conditions in matrix form, i.e., for the vector variable $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$ (use arrow notation for vectors!).

b) Use the eigenvector approach to find its general solution, showing all steps.

c) Find the IVP solution, showing all steps.

d) Make a single plot showing the three solution curves versus t on the same axes for at least 5 characteristic times of the slowest decaying exponential term in these expressions (i.e., in an appropriate viewing window).

e) Use calculus to find the local extrema for each of these three functions for $t > 0$ exactly (rules of exponents) and numerically giving both t and x_i to 3 significant digits. Are your results consistent with your plot?

f) Find the values of t after these extrema where each of these functions reaches one percent of its extreme value. Do these values seem consistent with your plot? Explain.

3. $x_1'(t) = x_1(t) - 4\,x_2(t),\; x_2'(t) = 2\,x_1(t) - 3\,x_2(t),\; x_1(0) = 1,\; x_2(0) = -1$.

a) Write this system and its initial conditions in matrix form, i.e., for the vector variable $\mathbf{x} = \langle x_1, x_2 \rangle$ (use arrow notation for vectors!).

b) Use the eigenvector approach to find its general solution in explicitly real form, showing all steps, and give formulas for the individual scalar variables.

c) Find the IVP solution, showing all steps, and give formulas for the individual scalar variables.

d) Express the sinusoidal factor in each solution function as a phase-shifted cosine.

e) Using your formulas from part c), make a single $t > 0$ plot for an interval about 5 times the characteristic time of the exponential factor, showing both solution curves and their exponentially decaying amplitude envelopes of the oscillations, annotating your paper printout to identify the individual solution curves by name. From the graph, estimate the time interval between the first peak of x_1 and the first peak of x_2 and convert it to an angle by multiplying by the frequency.

f) What is the difference in phase between these peaks in radians, degrees and in a fraction of a cycle, as calculated from the difference of your calculated phase shifts in part d)? Are these consistent with your estimate for the time interval between successive peaks in e) converted into an angle? Explain. Which of the two solution curves is ahead in time (has its peaks first)?

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: