MAT2705-01/06 07F Quiz 8 Print Name (Last, First) ______________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **You may use technology to find the roots of polynomials. [No mercy for mistakes here.]**

1. $y''+2y'+5y=\cos(\omega t)$, $\omega \ge 0$. [the last quiz found $y_h = e^{-t}\,(C1\cos(2t)+C2\sin(2t))$]

a) Use the method of undetermined coefficients to find the steady state sinusoidal solution (the particular solution $y_p$) of this driven harmonic oscillator DE.

b) Evaluate the amplitude $A(\omega)$ of this sinusoidal function. Show that it simplifies to the formula

$$A(\omega) = \left(25 - 6\,\omega^2 + \omega^4\right)^{-\frac{1}{2}}.$$

c) Use calculus to find the value $\omega_p$ at which the peak amplitude occurs, and evaluate $A(\omega_p)$, $A(0)$ and $A(\omega_p)/A(0)$, giving both their exact and numerical values.

d) How does the resonant frequency $\omega_p$ compare numerically to the natural frequency $\omega_0$ of the oscillator (i.e., what is their numerical ratio $\omega_p/\omega_0$)?

e) Use technology to plot $A(\omega)$ for $\omega = 0\,..\,\cancel{5}\ 10$. Sketch what you see, labeling the axes, tickmarks, variable values at the peak and the vertical intercept.

## ▶ solution

① a)

$$\begin{aligned}
5\,[\,y_p &= C_3\cos\omega t + C_4\sin\omega t\,]\\
2\,[\,y_p' &= -\omega C_3\sin\omega t + \omega C_4\cos\omega t\,]\\
1\,[\,y_p'' &= -\omega^2 C_3\cos\omega t - \omega^2 C_4\sin\omega t\,]
\end{aligned}$$

$$y_p''+2y_p'+5y_p = [(5-\omega^2)C_3 + 2\omega C_4]\cos\omega t + [-2\omega C_3 + (5-\omega^2)C_4]\sin\omega t = \cos\omega t$$

$$\begin{aligned}
(5-\omega^2)C_3 + 2\omega C_4 &= 1\\
-2\omega C_3 + (5-\omega^2)C_4 &= 0
\end{aligned}$$

$$\begin{bmatrix} 5-\omega^2 & 2\omega \\ -2\omega & 5-\omega^2 \end{bmatrix}\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 5-\omega^2 & 2\omega \\ -2\omega & 5-\omega^2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(5-\omega^2)^2+4\omega^2}\begin{bmatrix} 5-\omega^2 & -2\omega \\ 2\omega & 5-\omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(5-\omega^2)^2+4\omega^2}\begin{bmatrix} 5-\omega^2 \\ 2\omega \end{bmatrix}$$

$$\boxed{y_p = \frac{1}{\underbrace{(5-\omega^2)^2+4\omega^2}_{=\,25-10\omega^2+\omega^4+4\omega^2\,=\,25-6\omega^2+\omega^4\,=\,\mathcal{D}}}\left[(5-\omega^2)\cos\omega t + 2\omega\sin\omega t\right]}$$

b)

$$\begin{aligned}
A(\omega) &= \sqrt{C_3^2+C_4^2} = \sqrt{(5-\omega^2)^2+(2\omega)^2}\,/\,\mathcal{D}\\
&= \sqrt{(5-\omega^2)^2+4\omega^2}\,/\,\mathcal{D} = \sqrt{25-10\omega^2+\omega^4+4\omega^2}\,/\,\mathcal{D}\\
&= \sqrt{25-6\omega^2+\omega^4}\,/\,\mathcal{D} = \mathcal{D}^{1/2}/\mathcal{D} = \mathcal{D}^{-1/2}
\end{aligned}$$

$$\boxed{A(\omega) = \frac{1}{\sqrt{25-6\omega^2+\omega^4}} = \frac{1}{\sqrt{(5-\omega^2)^2+4\omega^2}}}$$

c)

$$\begin{aligned}
A'(\omega) &= \left((25-6\omega^2+\omega^4)^{-1/2}\right)'\\
&= -\tfrac{1}{2}(25-6\omega^2+\omega^4)^{-3/2}(0-12\omega+4\omega^3) = 0\\
&\rightarrow\ \omega^3-3\omega = (\omega^2-3)\omega = 0\\
&\omega = 0, \pm\sqrt{3} \xrightarrow[\omega>0]{} \boxed{\omega_p = \sqrt{3} \approx 1.73}
\end{aligned}$$

$$A(0) = 25^{-1/2} = \boxed{1/5} = 0.20$$

$$A(\omega_p) = A(\sqrt{3}) = (25-6\cdot 3+3^2)^{-1/2} = 16^{-1/2} = \boxed{\tfrac{1}{4}} = 0.25$$

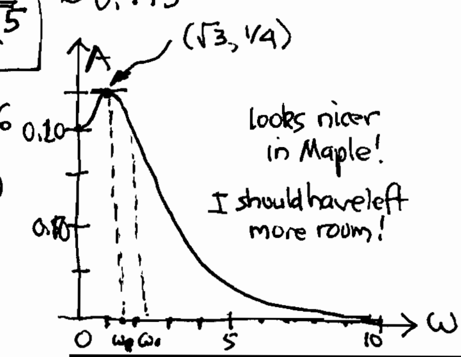
$$\frac{A(\omega_p)}{A(0)} = \frac{A(\sqrt{3})}{A(0)} = \frac{1/4}{1/5} = \boxed{\tfrac{5}{4}} = 1.25$$

d)

$$\frac{\omega_p}{\omega_0} = \boxed{\frac{\sqrt{3}}{\sqrt{5}}} \approx 0.775$$

$$\omega_0 \approx 2.236$$

e)




looks nicer in Maple!

I should have left more room!