MAT 2705-02/03 06S Final Exam Answers

1) a) $x'' + 20x' + 64x = 32\cdot 41\cos(20t)$

$x_h \sim e^{rt} \to r^2+20r+64=0$, $(r+4)(r+16)=0$

$r=-4,-16$

$e^{rt} = e^{-4t}, e^{-16t}$

$x_h = c_1e^{-4t}+c_2e^{-16t}$

$$64[\,x_p = c_3\cos 20t + c_4\sin 20t\,]$$
$$20[\,x_p' = -20c_3\sin 20t + 20c_4\cos 20t\,]$$
$$1[\,x_p'' = -400c_3\cos 20t - 400c_4\sin 20t\,]$$

$$x_p''+20x_p'+64x_p = [(64-400)c_3+400c_4]\cos 20t + [-400c_3+(64-400)c_4]\sin 20t = 32\cdot 41\cos 20t + 0\sin 20t$$

$$\begin{bmatrix}-336 & 400\\ -400 & -336\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}32\cdot 41\\ 0\end{bmatrix}$$

$$\left[\begin{array}{cc|c}-336 & 400 & 32\cdot 41\\ -400 & -336 & 0\end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & -21/13\\ 0 & 1 & 25/13\end{bmatrix} \qquad c_3=-21/13,\ c_4 = 25/13$$

$x_p = -\frac{21}{13}\cos 20t + \frac{25}{13}\sin 20t$

$$\boxed{x = x_h + x_p = c_1e^{-4t}+c_2e^{-16t} - \tfrac{21}{13}\cos 20t + \tfrac{25}{13}\sin 20t}$$

b) $x' = -4c_1e^{-4t} - 16c_2e^{-16t} + 20(\frac{21}{13})\sin 20t + 20(\frac{25}{13})\cos 20t$

$x(0) = c_1 + c_2 - 21/13 = 0$

$x'(0) = -4c_1 - 16c_2 + 20\cdot 25/13 = 64$

$$\begin{bmatrix}1 & 1\\ -4 & -16\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}21/13\\ 64-20\cdot 25/13\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & 21/13\\ -4 & -16 & 64-500/13\end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 167/39\\ 0 & 1 & -8/3\end{bmatrix} \qquad c_1 = 167/39,\ c_2 = -8/3$$

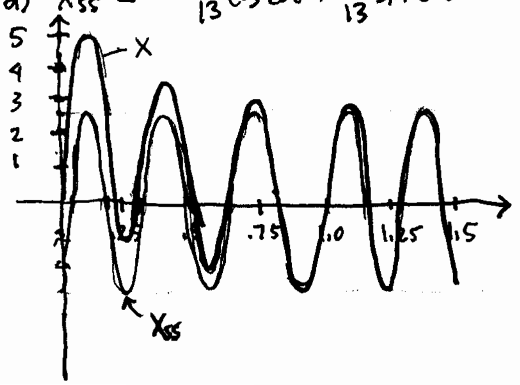
$$\boxed{\begin{aligned} x &= \tfrac{167}{39}e^{-4t} - \tfrac{8}{3}e^{-16t} + -\tfrac{21}{13}\cos 20t + \tfrac{25}{13}\sin 20t\\ &\approx 4.282e^{-4t} - 2.667e^{-16t} - 1.615\cos 20t + 1.923\sin 20t\end{aligned}}$$

c) $\tau_1 = \frac{1}{4} = .25$, $\tau_2 = \frac{1}{16} = .0625$ } self-explanatory

$T = \frac{2\pi}{\omega} = \frac{2\pi}{20} = \frac{\pi}{10} \approx 0.3146$

(BONUS: The second exponential decays away quickly compared to the first. The first exponential decays over several periods of the steady state solution.)

d) $x_{ss} = -\frac{21}{13}\cos 20t + \frac{25}{13}\sin 20t$



In about 4 or 5 characteristic times τ_1 (about $t = 1 \sim 1.25$) the transient dies away & you only see the steady state:

Amplitude $= \sqrt{21^2+25^2}/13 \approx 2.51$

2) a)
$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}' = \begin{bmatrix}-3 & 0 & 2\\ 3 & -10 & 0\\ 0 & 10 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \quad \begin{bmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 28\end{bmatrix}$$

b)
$$|A-\lambda I| = \begin{vmatrix}-3-\lambda & 0 & 2\\ 3 & -10-\lambda & 0\\ 0 & 10 & -2-\lambda\end{vmatrix} = -\lambda^3 - 15\lambda^2 - 56\lambda = -\lambda(\lambda^2+15\lambda+56) = -\lambda(\lambda+7)(\lambda+8) = 0$$

$\lambda = 0, -7, -8$

$$A - 0I = \begin{bmatrix}-3 & 0 & 2\\ 3 & -10 & 0\\ 0 & 10 & -2\end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{ccc}L & L & F\\ \hline 1 & 0 & -2/3\\ 0 & 1 & -1/5\\ 0 & 0 & 0\end{array} \qquad \begin{aligned}x_1 - \tfrac{2}{3}x_3 &= 0\\ x_2 - \tfrac{1}{5}x_3 &= 0\\ x_3 &= t\end{aligned}$$

$x_1 = \frac{2}{3}t$, $x_2 = \frac{1}{5}t$, $x_3 = t$

$\langle x_1, x_2, x_3\rangle = t\underbrace{\langle \tfrac{2}{3}, \tfrac{1}{5}, 1\rangle}_{\vec b_1}$

$$A + 7I = \begin{bmatrix}4 & 0 & 2\\ 3 & -3 & 0\\ 0 & 10 & 5\end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{ccc}L & L & F\\ \hline 1 & 0 & 1/2\\ 0 & 1 & 1/2\\ 0 & 0 & 0\end{array} \qquad \begin{aligned}x_1 + \tfrac{1}{2}x_3 &= 0\\ x_2 + \tfrac{1}{2}x_3 &= 0\\ x_3 &= t\end{aligned}$$

$x_1 = -\frac{1}{2}t$, $x_2 = -\frac{1}{2}t$, $x_3 = t$

$\langle x_1, x_2, x_3\rangle = t\underbrace{\langle -\tfrac{1}{2}, -\tfrac{1}{2}, 1\rangle}_{\vec b_2}$

$$A + 8I = \begin{bmatrix}5 & 0 & 2\\ 3 & -2 & 0\\ 0 & 10 & 6\end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{ccc}L & L & F\\ \hline 1 & 0 & 2/5\\ 0 & 1 & 3/5\\ 0 & 0 & 0\end{array} \qquad \begin{aligned}x_1 + \tfrac{2}{5}x_3 &= 0\\ x_2 + \tfrac{3}{5}x_3 &= 0\\ x_3 &= t\end{aligned}$$

$x_1 = -\frac{2}{5}t$, $x_2 = -\frac{3}{5}t$, $x_3 = t$

$\langle x_1, x_2, x_3\rangle = t\underbrace{\langle -\tfrac{2}{5}, -\tfrac{3}{5}, 1\rangle}_{\vec b_3}$

$$\underline{B} = \begin{bmatrix}\frac{2}{3} & -\frac{1}{2} & -\frac{2}{5}\\ \frac{1}{5} & -\frac{1}{2} & -\frac{3}{5}\\ 1 & 1 & 1\end{bmatrix} \qquad \vec{x}\,' = A\vec{x} \quad (\vec{x} = B\vec{y},\ \vec{y} = B^{-1}\vec{x})$$

$B^{-1}[(B\vec{y})' = A(B\vec{y})]$

$$\vec{y}\,' = (B^{-1}AB)\vec{y} = \begin{bmatrix}0 & 0 & 0\\ 0 & -7 & 0\\ 0 & 0 & -8\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} = \begin{bmatrix}0\\ -7y_2\\ -8y_3\end{bmatrix}$$

$$\left.\begin{array}{ll}y_1' = 0 & y_1 = c_1\\ y_2' = -7y_2 & y_2 = c_2e^{-7t}\\ y_3' = -8y_3 & y_3 = c_3e^{-8t}\end{array}\right\} \longrightarrow \vec{x} = B\vec{y}:$$

$$\boxed{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}\frac{2}{3} & -\frac{1}{2} & -\frac{2}{5}\\ \frac{1}{5} & -\frac{1}{2} & -\frac{3}{5}\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2e^{-7t}\\ c_3e^{-8t}\end{bmatrix} = \begin{bmatrix}\frac{2}{3}c_1 - \frac{1}{2}c_2e^{-7t} - \frac{2}{5}e^{-8t}\\ \frac{1}{5}c_1 - \frac{1}{2}c_2e^{-7t} - \frac{3}{5}e^{-8t}\\ c_1 + c_2e^{-7t} + c_3e^{-8t}\end{bmatrix}}$$

general soln.

$t=0$: $B\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 28\end{bmatrix} \to \langle B | \begin{bmatrix}0\\ 0\\ 28\end{bmatrix}\rangle \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0 & 15\\ 0 & 1 & 0 & 48\\ 0 & 0 & 1 & -35\end{bmatrix}$

$c_1 = 15$, $c_2 = 48$, $c_3 = -35$

$$\boxed{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}10 - 24e^{-7t} + 14e^{-8t}\\ 3 - 24e^{-7t} + 21e^{-8t}\\ 15 + 48e^{-7t} - 35e^{-8t}\end{bmatrix} \quad \text{\underline{IVP} solution}}$$

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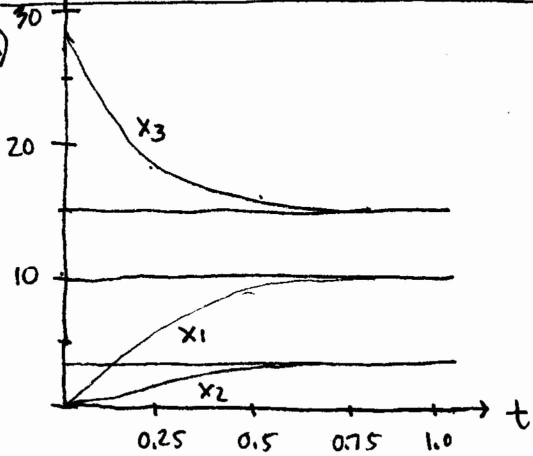
2 d)



e) $\lim_{t\to\infty} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 15 \end{bmatrix}$ exactly the 3 horizontal asymptotes!

3 a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -12 & 2 \\ 16 & -8 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b) $|A-\lambda I| = \begin{vmatrix} -12-\lambda & 2 \\ 16 & -8-\lambda \end{vmatrix} = (\lambda+8)(\lambda+12)-32 = \lambda^2+20\lambda+\underset{64}{\underline{96-32}}$
$= (\lambda+4)(\lambda+16) = 0$

$\lambda = -4, -16$

$A+4I = \begin{bmatrix} -8 & 2 \\ 16 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 - \frac{1}{4}x_2 = 0$, $x_2 = t$

$x_1 = \frac{1}{4}t$, $x_2 = t$ $\quad \langle x_1, x_2 \rangle = t \underbrace{\langle \tfrac{1}{4}, 1 \rangle}_{\vec{b}_1}$

$A+16I = \begin{bmatrix} 4 & 2 \\ 16 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 + \frac{1}{2}x_2 = 0$, $x_2 = t$

$x_1 = -\frac{1}{2}t$, $x_2 = t$ $\quad \langle x_1, x_2 \rangle = t \underbrace{\langle -\tfrac{1}{2}, 1 \rangle}_{\vec{b}_2}$

$B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}$, $\quad B^{-1} = \frac{4}{3}\begin{bmatrix} 1 & 1/2 \\ -1 & 1/4 \end{bmatrix}$

$\vec{x}'' = B\vec{x} \rightarrow \left(\begin{array}{l} \vec{x} = B\vec{y} \\ \vec{y} = B^{-1}\vec{x} \end{array} \right) \rightarrow B^{-1}(B\vec{y})'' = B^{-1}AB\vec{y}$
$\vec{y}'' = (B^{-1}AB)\vec{y}$

$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4y_1 \\ -16y_2 \end{bmatrix}$ $\quad y_1'' + 4y_1 = 0 \quad y_1 = c_1\cos 2t + c_2 \sin 2t$
$y_2'' + 16y_2 = 0 \quad y_2 = c_3\cos 4t + c_4 \sin 4t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1\cos 2t + c_2 \sin 2t \\ c_3 \cos 4t + c_4 \sin 4t \end{bmatrix} = \underbrace{(c_1\cos 2t + c_2\sin 2t)\begin{bmatrix} 1/4 \\ 1 \end{bmatrix}}_{\text{tandem mode}} + \underbrace{(c_3\cos 4t + c_4 \sin 4t)\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}_{\text{accordian mode}}$ gen soln

$= \begin{bmatrix} \frac{1}{4}(c_1\cos 2t + c_2 \sin 2t) - \frac{1}{2}(c_3 \cos 4t + c_4 \sin 4t) \\ (c_1 \cos 2t + c_2 \sin 2t) + (c_3 \cos 4t + c_4 \sin 4t) \end{bmatrix}$

c) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2c_1 \sin 2t + 2c_2 \cos 2t \\ -4c_3 \sin 4t + 4c_4 \cos 4t \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{4}{3}\begin{bmatrix} 1 & 1/2 \\ -1 & 1/4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2c_2 \\ 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} c_2 = 0 \\ c_4 = 0 \end{array}$

$$\boxed{\begin{array}{l} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\cos 2t \begin{bmatrix} 1/4 \\ 1 \end{bmatrix} - \cos 4t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{2}\cos 2t + \frac{1}{2}\cos 4t \\ 2\cos 2t - \cos 4t \end{bmatrix} \quad \text{IVP soln} \end{array}}$$

d) tandem mode: $\omega = 2$

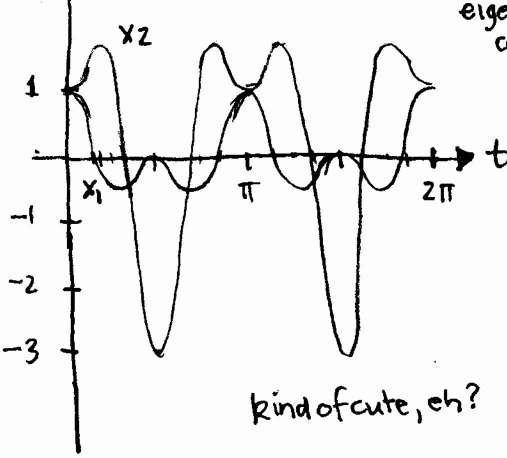
$\frac{x_1}{x_2} = \frac{1}{4}$ ratio of amplitudes

accordian mode: $\omega = 4$

$\frac{x_1}{x_2} = -\frac{1}{2}$ $\quad$ 180° out of phase
1:2 amplitude ratio.

The ratios of the amplitudes for a pure eigenvector solution at a single frequency are the same as the ratios of the abs values of the eigenvector components.

e)



kind of cute, eh?