MAT2705-01/04 05S Final Exam (1) Answers

① a) $\underbrace{x''+2x'+26x}_{\text{standard form}} = \underbrace{82\cos 4t - 82\sin 4t}_{\text{driving function}}$

$x_h \sim e^{rt}$: $r^2+2r+26=0$

$r = \frac{-2\pm\sqrt{4-4\cdot 26}}{2} = -1\pm 5i$

$e^{rt} = e^{(-1\pm 5i)t} = e^{-t}(\cos 5t \pm i\sin 5t)$

$\xrightarrow{\text{Re, Im}}$ $e^{-t}\cos 5t,\ e^{-t}\sin 5t$

$x_h = e^{-t}(c_1\cos 5t + c_2\sin 5t)$

$26\,[x_p = c_3\cos 4t + c_4\sin 4t]$

$2\,[x_p' = -4c_3\sin 4t + 4c_4\cos 4t]$

$1\,[x_p'' = -16c_3\cos 4t - 16c_4\sin 4t]$

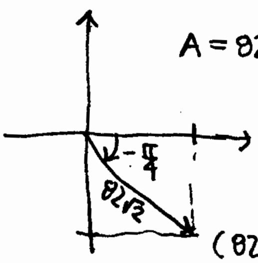
$x_p''+2x_p'+26x_p = [(26-16)c_3 + 8c_4]\cos 4t = 82\cos 4t$
$\quad + [-8c_3 + (26-16)c_4]\sin 4t \quad -82\sin 4t$

$10c_3 + 8c_4 = 82$
$-8c_3 + 10c_4 = -82$
$\begin{bmatrix} 10 & 8 & 82 \\ -8 & 10 & -82 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -1 \end{bmatrix}$ $c_3 = 9$, $c_4 = -1$

$x_p = 9\cos 4t - \sin 4t$ response function

$$\boxed{x = x_h + x_p = \underbrace{e^{-t}(c_1\cos 5t + c_2\sin 5t)}_{\text{transient (decays away)}} + \underbrace{9\cos 4t - \sin 4t}_{\text{steady state (remains)}}}$$

b) $82\cos 4t - 82\sin 4t = \boxed{82\sqrt{2}\cos(4t - \pi/4)}$

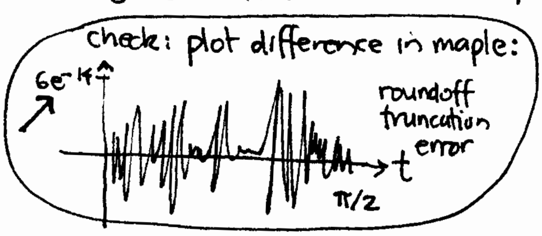


$A = 82\sqrt{1^2+(-1)^2} = 82\sqrt{2}$

$\tan\delta = \frac{-82}{82} = -1$

$\delta = \arctan(-1) = -\arctan 1 = -\pi/4$

check: plot difference in maple:



a) now impose initial conditions on general soln:

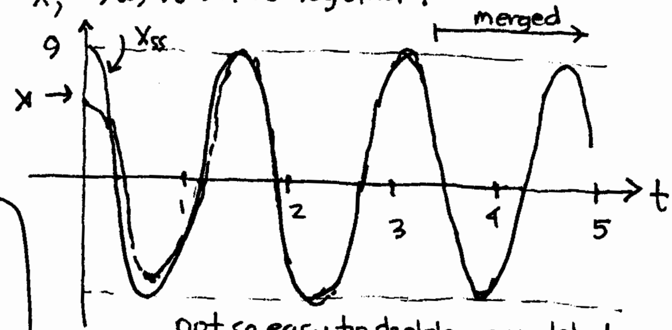
$x' = -e^{-t}(c_1\cos 5t + c_2\sin 5t) \quad -36\sin 4t - 4\cos 4t$
$\quad + e^{-t}(-5c_1\sin 5t + 5c_2\cos 5t)$

$x(0) = c_1 + 9 = 5 \to c_1 = -4$

$x'(0) = -c_1 + 5c_2 - 4 = 0 \to c_2 = \frac{4+(-4)}{5} = 0$

$$\boxed{x = -4e^{-t}\cos 5t + 9\cos 4t - \sin 4t}$$

c) plot x, $9\cos 4t - \sin 4t$ together:



not so easy to sketch accurately!

② a) $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = -(\lambda-3)^2(\lambda-5) = 0$

$\boxed{\lambda = 3, 3, 5}$

b) $\lambda = 3$: $A - 3I = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{ccc} L & F & F \\ \hline \end{array} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + x_2 - x_3 = 0 \to x_1 = -t_1 + t_2$

$x_2 = t_1$

$x_3 = t_2$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_1 + t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{E_1} + t_2 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{E_2}$

$\boxed{\{E_1, E_2\} = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \text{ is a basis of the } \lambda = 3 \text{ eigenspace}}$

② c)

$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+2-1 \\ -2+10-2 \\ -1+2+2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 3 \end{bmatrix}$

$= 3\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ always factor out common factors

Therefore this is an eigenvector with eigenvalue 3 and so can be expressed in terms of the basis $\{E_1, E_2\}$:

d) $c_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ system in standard form

$\begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $c_1 = 2$, $c_2 = 1$

$$\boxed{[-1, 2, 1] = 2[-1, 1, 0] + [1, 0, 1]}$$

MAT2705-01/04 05S Final Exam Answers (2)

3 a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\underline{x}'' = \underline{A}\ \underline{x}$

b) $\det(A-\lambda I) = \begin{vmatrix} -10-\lambda & 6 \\ 6 & -10-\lambda \end{vmatrix} = (\lambda+10)^2 - 36$

$= \lambda^2 + 20\lambda + 64 = (\lambda+4)(\lambda+16) = 0$

$\lambda = -4, -16$

c) $\lambda = -4$: $\underline{A} + 4\underline{I} = \begin{bmatrix} -10+4 & 6 \\ 6 & -10+4 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix}$

$\xrightarrow{\text{rref}} \begin{array}{cc} L & F \\ \end{array}$ $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 - x_2 = 0$, $x_2 = t$, $x_1 = t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{E_1}$

$\lambda = -16$: $\underline{A} + 16\underline{I} = \begin{bmatrix} -10+16 & 6 \\ 6 & -10+16 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \longrightarrow$

$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 + x_2 = 0 \rightarrow x_1 = -t$, $x_2 = t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{E_2}$

d) $\underline{B} = \text{augment}(\underline{E}_1, \underline{E}_2) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\underline{B}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\underline{B}^{-1}\underline{A}\,\underline{B} = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix} = \underline{A}_B$

$\underline{y}'' = \underline{A}_B\underline{y}$ $\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4y_1 \\ -16y_2 \end{bmatrix}$

$y_1'' = -4y_1 \rightarrow y_1'' + 4y_1 = 0 \longrightarrow \boxed{y_1 = c_1\cos 2t + c_2 \sin 2t}$

$y_2'' = -16y_2 \rightarrow y_2'' + 16y_2 = 0 \longrightarrow \boxed{y_2 = c_3\cos 4t + c_4\sin 4t}$

$y'' + \omega^2 y = 0 \longrightarrow y = a\cos\omega t + b\sin\omega t$

$\left(y \sim e^{rt} \rightarrow r^2 + \omega^2 = 0,\ r = \pm i\omega \quad e^{\pm i\omega t} = \cos\omega t \pm i\sin\omega t \xrightarrow[\text{Im}]{\text{Re}} \cos\omega t, \sin\omega t \right)$

e) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\cos 2t + c_2\sin 2t \\ c_3\cos 4t + c_4\sin 4t \end{bmatrix}$

$= \begin{bmatrix} c_1\cos 2t + c_2\sin 2t - c_3\cos 4t - c_4\sin 4t \\ c_1\cos 2t + c_2\sin 2t + c_3\cos 4t + c_4\sin 4t \end{bmatrix}$

f) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2c_1\sin 2t + 2c_2\cos 2t + 4c_3\sin 4t - 4c_4\cos 4t \\ -2c_1\sin 2t + 2c_2\cos 2t - 4c_3\sin 4t + 4c_4\cos 4t \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 - c_3 \\ c_1 + c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 2c_2 - 4c_4 \\ 2c_2 + 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{bmatrix}$ $c_1 = 1/2$, $c_3 = -1/2$

$\begin{bmatrix} 2 & -4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 & -4 & 0 \\ 2 & 4 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $c_2 = 0$, $c_4 = 0$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cos 2t + \frac{1}{2}\cos 4t \\ \frac{1}{2}\cos 2t - \frac{1}{2}\cos 4t \end{bmatrix} = \frac{1}{2}\cos 2t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\cos 4t\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

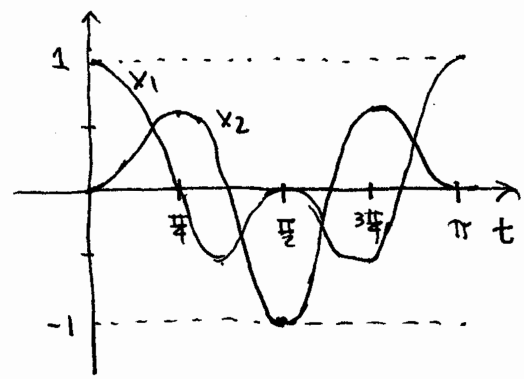
slow (yoyo) mode   fast (accordian) mode

g) x_1: $\omega_1 = 2$, $T_1 = 2\pi/2 = \pi$ ← smallest common period

x_2: $\omega_2 = 4$ $T_2 = 2\pi/4 = \pi/2$

↳ repeats twice for each x_1 oscillation

(just trial & error plot to see period!)



In the diagram spring 1 is stretched pulling mass 1 left while spring 2 is compressed, pushing mass 1 left and mass 2 right, which is what this graph shows.

However, on the saturday exam I had mistakenly made $x_1(0) = 0$, $x_2(0) = 1$ when copying it over, without changing the diagram. so this switches the solution functions and should have conflicted with your intuition from the diagram.