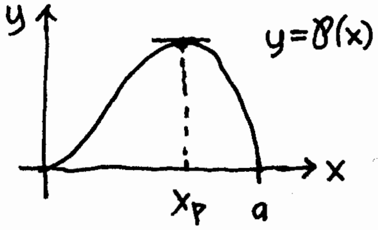
MAT1505-05 04S Quiz 6 Print Name (Last, First) ______________________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (not decimal approximations, if possible).



The probability distribution $\mathcal{B}(x) = c\,(a-x)\,x^2 = c\,(ax^2-x^3)$ is a beta-distribution for $0 \le x \le a$.

a) For what value of c is the total probability equal to 1 : $P(0 \le x \le a) = \int_0^a \mathcal{B}(x)\,dx = 1$? Set c equal to this value for the rest of the problem.

b) Use calc1 techniques to determine the value x_p at which this distribution has its peak value.

c) Evaluate the expected value μ of the variable x : $\mu = \int_0^a x\,\mathcal{B}(x)\,dx$.

d) By changing the independent variable from x to $u = x/a$ (dimensionless!) on the interval $0 \le u \le 1$, show that

$$P(x_1 \le x \le x_2) = \underbrace{\int_{x_1}^{x_2} \mathcal{B}(x)\,dx}_{\text{i.e. do a u-sub here to get this result}} = \int_{u_1}^{u_2} 12(1-u)u^2\,du = P(u_1 \le u \le u_2),\ \text{where } u_i = x_i/a.$$

e) This enables us to find the median x_m of the distribution by finding $u_m = x_m/a$ by solving $\int_0^{u_m} 12(1-u)u^2\,du = \frac{1}{2}$. The resulting quartic equation is easily solved for $0 \le u_m \le 1$ on your graphing calculator or with Maple : > fsolve (eq, u = 0..1); What is u_m to 3 decimal places? How does it compare to the peak value $u_p = x_p/a$ and expected value $u_e = \mu/a$? (order them : A < B < C by decimal values).

a) $1 = \int_0^a c\,(ax^2-x^3)\,dx = c\left(a\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^a = c\left(a\frac{a^3}{3} - \frac{a^4}{4}\right) = ca^4\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{ca^4}{12} \to \boxed{c = \frac{12}{a^4}}$

b) $\mathcal{B}'(x) = \frac{d}{dx}\left[\frac{12}{a^4}(ax^2-x^3)\right] = \frac{12}{a^4}(2ax-3x^2) = \frac{12x}{a^4}(2a-3x) = 0 \to x = \frac{2}{3}a, 0 \to \boxed{x_p = \frac{2a}{3}}$

c) $\mu = \int_0^a x \cdot \frac{12}{a^4}(ax^2-x^3)\,dx = \frac{12}{a^4}\int_0^a (ax^3-x^4)\,dx = \frac{12}{a^4}\left[\frac{ax^4}{4} - \frac{x^5}{5}\right]\Big|_0^a = \frac{12}{a^4}a^5\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{12}{20}a = \boxed{\frac{3}{5}a = \mu}$

d) $\int_{x_1}^{x_2} \frac{12}{a^4}(ax^2-x^3)\,dx = \int_{u_1}^{u_2} \frac{12}{a^4}\left(a(ua)^2 - (ua)^3\right)(a\,du) = \int_{u_1}^{u_2} \frac{12}{a^3}(a^3u^2 - a^3u^3)\,du$

$u = x/a \to x = ua$

$\frac{du}{dx} = 1/a$

$du = dx/a$

$= \int_{u_1}^{u_2} 12(u^2-u^3)\,du = \int_{u_1}^{u_2} 12(1-u)u^2\,du$ ✓

e) $\frac{1}{2} = \int_0^{u_m} 12(u^2-u^3)\,du = 12\left(\frac{u^3}{3} - \frac{u^4}{4}\right)\Big|_0^{u_m} = \boxed{4u_m^3 - 3u_m^4 = \frac{1}{2}} \to 0.6142724319 \sim \boxed{0.614 = u_m}$

$u_p = \frac{x_p}{a} = \frac{2}{3} \sim \underline{0.667}$, $u_e = \frac{\mu}{a} = \frac{3}{5} = \underline{0.600}$ so $\boxed{u_e < u_m < u_p}$