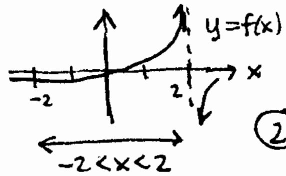
# MAT 1505-05 04S Test 3 (TakeHome) Answers

① a) $f(x)=\dfrac{x}{8(1-\frac{x^3}{8})}=\dfrac{x}{8}\cdot\dfrac{1}{1-(\frac{x^3}{8})}=\dfrac{x}{8}\sum_{n=0}^{\infty}\left(\frac{x^3}{8}\right)^n=\sum_{n=0}^{\infty}\dfrac{x}{8}\dfrac{x^{3n}}{8^n}=\boxed{\sum_{n=0}^{\infty}\dfrac{x^{3n+1}}{8^{n+1}}}$

valid for $|r|=|\frac{x^3}{8}|<1$ $\longrightarrow$ $\frac{|x^3|}{8}<1 \to |x^3|<8 \to |x|^3<8 \to \boxed{|x|<2}$

b)




The function has a vertical asymptote at $x=2$, forcing its series to diverge there. Yes, consistent.

② a) $a_n=\dfrac{(-1)^n x^n}{n^2 5^n}$ $\quad\left|\dfrac{a_{n+1}}{a_n}\right|=\dfrac{|x|^{n+1}}{(n+1)^2 5^{n+1}}\Big/\dfrac{|x|^n}{n^2 5^n}=\dfrac{|x|^{n+1}}{|x|^n}\dfrac{5^n}{5^{n+1}}\dfrac{n^2}{(n+1)^2}$

$=\dfrac{|x|}{5}\left(\dfrac{n}{n+1}\right)^2$ $\quad\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\dfrac{|x|}{5}\left(\dfrac{n}{n+1}\right)^2=\dfrac{|x|}{5}<1$ for convergence $\to \boxed{|x|<5=R}$

endpoints $x=5$: $\sum_{n=1}^{\infty}(-1)^n\dfrac{5^n}{n^2 5^n}=\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n^2}$ alternating series of decreasing (abs val) terms so converges by alt series test

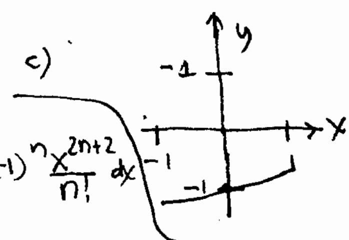
$x=-5$: $\sum_{n=1}^{\infty}(-1)^n\dfrac{(-5)^n}{n^2 5^n}=\sum_{n=1}^{\infty}\underbrace{(-1)^n(-1)^n}_{+1}\dfrac{5^n}{n^2 5^n}=\sum_{n=1}\dfrac{1}{n^2}$ $\quad p=2$ series, $p>1$ so converges

so the complete interval of convergence includes the endpoints: $\boxed{-5\le x\le 5}$

③ a) $\lim_{x\to 0}\dfrac{1-\cos x}{1+x-e^x}=\lim_{x\to 0}\dfrac{1-(1-\frac{x^2}{2}+\frac{x^4}{24}-\dots)}{1+x-(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots)}=\lim_{x\to 0}\dfrac{\frac{x^2}{2}-\frac{x^4}{24}+\dots}{-\frac{x^2}{2}-\frac{x^3}{6}-\dots}=\lim_{x\to 0}\dfrac{\frac{1}{2}-\frac{x^2}{12}+\dots}{-\frac{1}{2}-\frac{x}{6}-\dots}$

$=\lim_{x\to 0}\dfrac{\frac{1}{2}}{-\frac{1}{2}}=\boxed{-1}$ $\quad$ b) $\lim_{x\to 0}\dfrac{1-\cos x \to 1-1=0}{1+x-e^x \to 1-1=0}\sim\dfrac{0}{0}\to=\lim_{x\to 0}\dfrac{0-(-\sin x)}{0+1-e^x}=\lim_{x\to 0}\dfrac{\sin x\to 0}{1-e^x\to 1-1=0}\sim\dfrac{0}{0}\to$

$=\lim_{x\to 0}\dfrac{\cos x}{0-e^x}=\lim_{x\to 0}\dfrac{\cos x}{-e^x}=\dfrac{\cos(0)}{-e^0}=\dfrac{1}{-1}=\boxed{-1}$ $\quad$ c)




the graph clearly has $y=-1$ as its y intercept, consistent with limit.

④ a) $\int_0^{0.5}x^2e^{-x^2}dx=\int_0^{0.5}x^2\left(\sum_{n=0}^{\infty}\dfrac{(-x^2)^n}{n!}\right)dx=\int_0^{0.5}\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+2}}{n!}dx$

$=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+3}}{(2n+3)n!}\Big|_0^{0.5}=\sum_{n=0}^{\infty}(-1)^n\dfrac{(0.5)^{2n+3}}{(2n+3)n!}$ $\quad$ alternating series of decreasing (abs val) terms so must converge

$=\dfrac{(.5)^3}{3}-\dfrac{(.5)^5}{5}+\dfrac{(.5)^7}{7\cdot 2!}-\dfrac{(.5)^9}{9\cdot 3!}+\dots=.0461667-.0062500+.0005580\,|\,-\underbrace{.0000362}_{<.5\times 10^{-3}}+\dots$

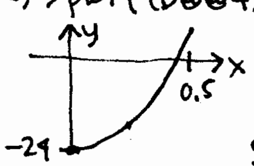
first 3 terms give desired accuracy $\quad\boxed{0.035975}\boxed{\sim 0.036}$

b) `> evalf(Int(x^2*exp(-x^2), x=0..0.5));` $\quad$ 0.035940 $\quad$ error: $0.035940-0.035975=\boxed{-.000035}$

Yes, consistent with error goal & the next term in the series. $\quad$ smaller than

c) `> f := x -> x^2 * exp(-x^2);`
`> plot((D@@4)(f)(x), x=0..0.5);`

$|E_S|<\dfrac{K(b-a)^5}{180n^4}=\dfrac{24(.5)^5}{180\cdot 2^4}=.00026<\frac{1}{2}\times 10^{-3}$




so take $K=24$ in error formula

also gives 3 decimal place accuracy! $\quad n=2\to\Delta x=1/4$

$S_2=\frac{1}{3}(\frac{1}{4})(f(0)+4f(.25)+f(.5))=\boxed{0.035796}\boxed{\sim 0.036}$

⑤ a) $f(x)=(1+x)^{-2}$ $\quad f(0)=1$

$f'(x)=-2(1+x)^{-3}$ $\quad f'(0)=-2$

$f''(x)=(-2)(-3)(1+x)^{-4}$ $\quad f''(0)=+2\cdot 3$

$f^{(3)}(x)=(-2)(-3)(-4)(1+x)^{-5}$ $\quad f^{(3)}(0)=-2\cdot 3\cdot 4$

$S_4=1-2x+\dfrac{2\cdot 3}{2!}x^2-\dfrac{2\cdot 3\cdot 4}{3!}x^3+\dots]$

$=\boxed{1-2x+3x^2-4x^3+\dots}$ obvious!

b) $=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

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⑤ c) $(1+x)^{-2} = -\frac{d}{dx}(1+x)^{-1} = -\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^{n+1} n x^{n-1}$

$m = n-1 \rightarrow n = m+1$

$n=0 \rightarrow 1$ (since term 0 when $n=0$)

$= \sum_{m=0}^{\infty} \underbrace{(-1)^{m+2}}_{(-1)^m \underbrace{(-1)^2}_{+1}} (m+1)x^m = \sum_{m=0}^{\infty}(-1)^m (m+1)x^m = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ same as before

d) $F = \frac{mgR^2}{(R+h)^2} = \frac{mg}{(R+h)^2/R^2} = \frac{mg}{\left(\frac{R+h}{R}\right)^2} = \frac{mg}{(1+\frac{h}{R})^2} = mg\left(1+\frac{h}{R}\right)^{-2} = mg\, f\left(\frac{h}{R}\right)$

$\approx mg\left(1 - 2\left(\frac{h}{R}\right) + \cdots\right) = \underbrace{mg}_{\text{first term}} - \underbrace{2mg\frac{h}{R}}_{\text{second term}}$

ratio: $\frac{2mgh/R}{mg} = .01 \rightarrow 2\left(\frac{h}{R}\right) = .01 \rightarrow \frac{h}{R} = .005 \rightarrow$ $\boxed{\begin{array}{l} h = .005R \\ \approx 31.855 \text{ km} \\ \approx 32 \text{ km} \\ \approx 20 \text{ miles} \end{array}}$

above this height the error compared to the first term exceeds 1% of the first term.