MATH505 04S Test 2 Answers

① a) Division by zero occurs in the integrand at $x=2$ at the right endpoint of the interval of integration.

b) $\int_0^2 \frac{x}{(4-x^2)^{1/2}}dx = \lim_{a\to 2^-}\int_0^a \frac{x}{(4-x^2)^{1/2}}dx \leftarrow \int \frac{\overbrace{(xdx)}^{=-\frac{du}{2}}}{(4-x^2)^{1/2}} = -\frac{1}{2}\int u^{-1/2}du = -\frac{1}{2}\frac{u^{1/2}}{(1/2)}+c$

$= \lim_{a\to 2^-} -(4-x^2)^{1/2}\Big|_0^a = \lim_{a\to 2^-}\left[-(4-a^2)^{1/2}+2\right]$ $\qquad = -(4-x^2)^{1/2}+c$

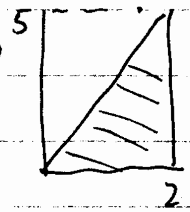
$u = 4-x^2$

$\frac{du}{dx} = -2x \quad du = -2xdx$

$-\frac{du}{2} = xdx$

$= 0+2 = \boxed{2}$

c) $u^{-3/2} \longrightarrow u^{-1/2} = (4-x^2)^{-1/2}$ still have division by zero in antiderivative, limit will go to infinity.

② a) 

triangle area $= \frac{1}{2}(2)(5) = 5$ roughly similar to area under curve

so $f_{avg} \approx 5/2 = 2.5$

b) $\int_0^2 x^3\sqrt{4-x^2}\,dx = \int_{x=0}^{x=2} \underbrace{x^2}_{4-u}\underbrace{(4-x^2)^{1/2}}_{u^{1/2}}\underbrace{x\,dx}_{-\frac{du}{2}} = \int_4^0 (4-u)u^{1/2}\left(\frac{du}{-2}\right)$

$u = 4-x^2 \rightarrow x^2 = 4-u$

$du = -2xdx$

$x=0 \rightarrow u=4$

$x=2 \rightarrow u=0$

$= -\frac{1}{2}\int_4^0 (4-u)u^{1/2}du$

$= \boxed{\frac{1}{2}\int_0^4 (4-u)u^{1/2}du}$

c) $= \frac{1}{2}\int_0^4 4u^{1/2}-u^{3/2}\,du = \frac{1}{2}\left[4\left(\frac{2u^{3/2}}{3}\right)-\frac{2}{5}u^{5/2}\right]_0^4 = \frac{2}{2}\left[\frac{4}{3}4^{3/2}-\frac{1}{5}4^{5/2}\right] = 4^{5/2}\left(\frac{1}{3}-\frac{1}{5}\right)$

$2^5 = 32 \quad 2/15$

$= \frac{64}{15} \longrightarrow f_{avg} = \frac{1}{2}\int_0^2 f(x)dx = \frac{1}{2}\left(\frac{64}{15}\right) = \boxed{\frac{32}{15}} \approx 2.13$

not so far from our estimate.

③ $6\text{ min} = \frac{6\text{ min}(1\text{ hr})}{60\text{ min}} = \frac{1}{10}\text{ hr} \qquad \left(50\frac{\text{mi}}{\text{hr}}\right)\left(\frac{1}{10}\text{ hr}\right) = 5\text{ mi}$ (distance should be roughly like this)

$\Delta t = 1\text{ min} = \frac{1}{60}\text{ hr} \qquad \text{distance} = \int_0^{1/10} v(t)dt \approx \frac{1}{60}\cdot\frac{1}{3}\left[v_0+4v_1+2v_2+4v_3+2v_4+4v_5+v_6\right]$

$= \frac{1}{180}\left[40+4(42)+2(45)+4(49)+2(52)+4(54)+56\right] = 4.83 \approx \boxed{4.8\text{ mi}}$

④ a) $\int \underbrace{k^2x}_{u}\underbrace{e^{-kx}dx}_{dv} = k^2x\left(\frac{e^{-kx}}{-k}\right) - \underbrace{\int\left(\frac{e^{-kx}}{-k}\right)k^2dx}_{+k\int e^{-kx}dx} = \boxed{\begin{array}{l}-kxe^{-kx}-e^{-kx}+C \\ = -(kx+1)e^{-kx}+C\end{array}}$

$\left(\frac{e^{-kx}}{-k}\right)$

$u = k^2x \quad dv = e^{-kx}dx$

$du = k^2dx \quad v = \frac{e^{-kx}}{-k}$

b) $\int_0^\infty k^2xe^{-kx}dx = \lim_{a\to\infty}\int_0^a k^2xe^{-kx}dx = \lim_{a\to\infty} -(kx+1)e^{-kx}\Big|_0^a$

$= \lim_{a\to\infty}\left[-(ka+1)e^{-ka}+e^0\right] = \lim_{a\to\infty}\left[1-\underbrace{\left(\frac{ka+1}{e^{ka}}\right)}_{\to 0}\right] = 1$ ✓

$\left(\lim_{a\to\infty}\frac{ka+1}{e^{ka}} = \lim_{a\to\infty}\frac{k}{e^{ka}k} = \lim_{a\to\infty}\frac{1}{e^{ka}} = 0\right)$

exponentials beat polynomials or L'Hopital's rule

c) $0 = P'(x) = \frac{d}{dx}\left(k^2xe^{-kx}\right) = k^2\left[1e^{-kx}+xe^{-kx}(-k)\right]$

$= k^2e^{-kx}(1-kx) \rightarrow 1-kx = 0 \rightarrow kx = 1$

$x = \boxed{1/k = x_p}$

d) $\int_0^{1/k} k^2xe^{-kx}dx \overset{\text{part a)}}{=} -(kx+1)e^{-kx}\Big|_0^{1/k} = -\left(k\cdot\frac{1}{k}+1\right)e^{-k/k}+1e^0$

$= \boxed{1-2e^{-1} \approx 0.264}$