MAT1500-04/06 04F Final Exam Answers

1 a) $f(x)=\ln(1+x)$

$f'(x)=\frac{d}{dx}\ln(1+x)=\frac{1}{1+x}\frac{d}{dx}\overbrace{(1+x)}^{0+1}=\frac{1}{1+x}$

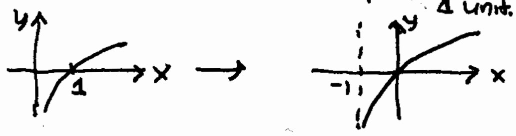
$f(0)=\ln 1=0 \leftrightarrow (x,y)=(0,0)$

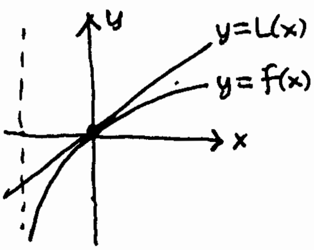
$f'(0)=\frac{1}{1+0}=1 \rightarrow m=1$

$y-0=1(x-0)\rightarrow y=\boxed{x=L(x)}$

(or $L(x)=f(0)+f'(0)(x-0)=0+1(x-0)$
$=x$)

b) $f(x)=\ln(\underbrace{x+1})$
add 1 to input, shift graph left 1 unit.





linear approx too high: f is concave down so lies below tangent line except at pt of tangency:

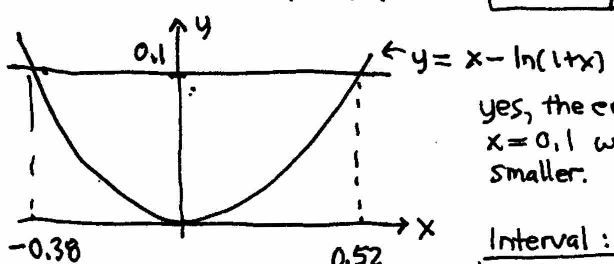
$f''(x)=\frac{d}{dx}(x+1)^{-1}=-(x+1)^{-2}(1+0)$

$=-\frac{1}{(x+1)^2}<0$ for $x+1>0$ concave down

c) $\ln 1.1=f(.1)$

$\approx L(.1)=\boxed{.1}$ Error $=L(.1)-\ln(1.1)$
$=.1-.0953=\boxed{0.0047}$

d)



yes, the error at $x=0.1$ was much smaller.

Interval: $\boxed{-0.38<x<0.52}$

2 a) $h=6\text{in}$, $\frac{dr}{dt}=\frac{.001\text{ in}}{3\text{ min}}$, $\left.\frac{dV}{dt}\right|_{r=3.800/2}=?$

$V=\pi r^2h=6\pi r^2$

$\frac{dV}{dt}=6\pi(2r\frac{dr}{dt})=12\pi r\frac{dr}{dt}$

$\left.\frac{dV}{dt}\right|_{r=3.800/2}=12\pi\left(\frac{3.800}{2}\right)\left(\frac{.001}{3}\right)=2\pi(3.8)(.001)$

$=\boxed{0.0239\ \text{in}^3/\text{min}}$

b) $r=3.800/2$, $dr=.005$

$\frac{dV}{dr}=\frac{d}{dr}(6\pi r^2)=12\pi r\rightarrow dV=12\pi r\,dr$

$dV|_{r=3.8/2,\ dr=.005}=12\pi\left(\frac{3.8}{2}\right)(.005)=\boxed{0.358\ \text{in}^3}$

$\frac{dV}{V}=\frac{12\pi r\,dr}{6\pi r^2}=2\frac{dr}{r}$, $\frac{dr}{r}=\frac{.005}{3.8/2}=0.00263\rightarrow\boxed{0.26\%}$

$\frac{dV}{V}\rightarrow\boxed{0.53\%}$

5 a) $y=f(x)=(1-x^2)e^{-\frac{x}{2}}=0\rightarrow x=\pm1$

$x=0\rightarrow y=f(0)=1e^0=1$

$\lim_{x\to\infty}\underbrace{(1-x^2)}_{-\infty}\underbrace{e^{-\frac{x}{2}}}_{e^{-\infty}=0}=\lim_{x\to\infty}\frac{1-x^2\to-\infty}{e^{x/2}\to\infty}\overset{?}{=}\lim_{x\to\infty}\frac{-2x}{e^{\frac{x}{2}}\cdot\frac12}$

$\overset{?}{=}\lim_{x\to\infty}\frac{-4x\to\infty}{e^{x/2}\to\infty}=\lim_{x\to\infty}\frac{-4}{e^{x/2}\cdot\frac12}=-\frac{8}{\infty}=\boxed{0}\rightarrow$

$\boxed{y=0}$ H-asymptote

$\lim_{x\to-\infty}\underbrace{(1-x^2)}_{-\infty}\underbrace{e^{-x/2}}_{e^{\infty}=\infty}=\boxed{-\infty}$

b) $f(x)=(1-x^2)e^{-\frac{x}{2}}$

$f'(x)=\frac{d}{dx}((1-x^2)e^{-x/2})=\overbrace{\frac{d}{dx}(1-x^2)}^{0-(2x)}e^{-\frac{x}{2}}+(1-x^2)\overbrace{\frac{d}{dx}e^{-\frac{x}{2}}}^{e^{-\frac{x}{2}}(-\frac12)}$

$=-2xe^{-\frac{x}{2}}-\frac12(1-x^2)e^{-x/2}=\boxed{\left(\frac{x^2-1-4x}{2}\right)e^{-\frac{x}{2}}}$

$f''(x)=\frac12\left[\frac{d}{dx}(x^2-1-4x)e^{-\frac{x}{2}}+(x^2-1-4x)\frac{d}{dx}(e^{-\frac{x}{2}})\right]$

$=\frac12\left[(2x-0-4)e^{-\frac{x}{2}}+(x^2-1-4x)e^{-\frac{x}{2}}(-\tfrac12)\right]$

$=\frac14e^{-\frac{x}{2}}(4x-8-x^2+1+4x)$ sign:

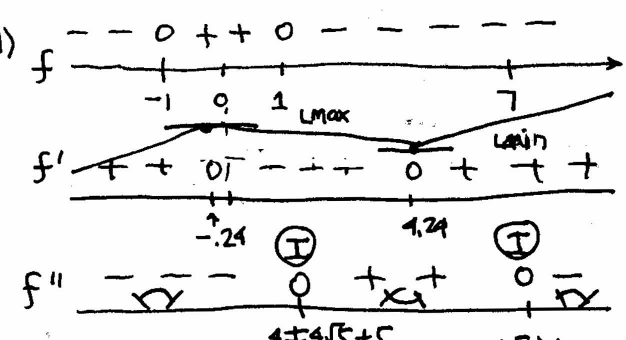
$=-\frac14e^{-\frac{x}{2}}(x^2-8x+7)=\boxed{-\frac14e^{-\frac{x}{2}}(x-1)(x-7)}$

c) $0=f'(x)\rightarrow x^2-4x-1=0\rightarrow$ sign: + − +

$x=\frac{4\pm\sqrt{16-4(-1)}}{2}=\boxed{2\pm\sqrt5\approx-0.236,4.236}$

$0=f''(x)\rightarrow x=\boxed{1,7}$

d)



e) $f(2\pm\sqrt5)=(1-(2\pm\sqrt5)^2)e^{-(2\pm\sqrt5)/2}$

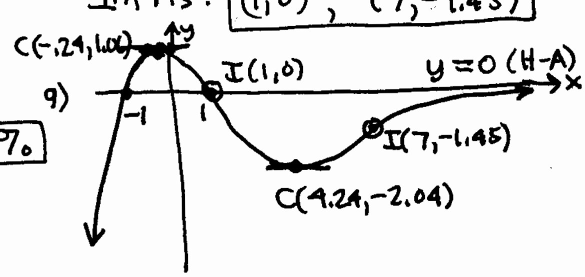
$-4(2\pm\sqrt5)!=(-8\mp4\sqrt5)\quad\approx-2.04,1.06$

Crits: $\boxed{(-0.24,1.06),(4.24,-2.04)}$

f) $f(1)=0$, $f(7)=(1-49)e^{-7/2}=-48e^{-7/2}$
≈-1.45

Infl Pts: $\boxed{(1,0),\ (7,-1.45)}$

g)



3a) $A = (\omega^4 - 16\omega^2 + 256)^{-1/2}$

$$\frac{dA}{d\omega} = -\frac{1}{2}(\omega^4 - 16\omega^2 + 256)^{-3/2}(4\omega^3 - 16(2\omega) + 0)$$

$$= \frac{2\omega(8-\omega^2)}{(\omega^4-16\omega^2+256)^{3/2}} = 0 \to \omega = 0, \quad \boxed{\omega = \sqrt{8} = 2\sqrt{2} \approx 2.83}$$

b)

$\frac{dA}{d\omega}$: 0 ++ 0 − − −

$0 \qquad 2\sqrt{2} \qquad \to x$

Increasing from 0 to $2\sqrt{2}$, decreasing after $\to$ global maximum.

$$A(2\sqrt{2}) = \frac{1}{(8^2 - 16\cdot 8 + 256)^{1/2}} = \boxed{\frac{1}{\sqrt{192}} = \frac{1}{8\sqrt{3}}} \approx \boxed{.0722}$$

2b) alternate solution given part a)

$$dr = \frac{dr}{dt}\,dt$$

$$.005 \text{ in} = \frac{.001 \text{ in}}{3 \text{ min}}\,dt$$

$$dt = 15 \text{ min}$$

$$dV = \frac{dV}{dt}\,dt$$

$$= \left(.0239\,\frac{\text{in}^3}{\text{min}}\right)(15\text{ min})$$

$$= \boxed{0.358 \text{ in}^3}$$

4) x = first #, y = second #

$0 \le x \le 1, \quad 0 \le y \le 1$

$x + y = 1$ (constraint) $\to y = 1 - x$

$F = x^3 + 4y^3 = x^3 + 4(1-x)^3$

$f(x) = x^3 + 4(1-x)^3$ (maximize, minimize) continuous differentiable function on closed interval

$$f'(x) = 3x^2 + 4(3)(1-x)^2(0-1)$$
$$= 3x^2 - 12(1-x)^2$$
$$= 3[x^2 - 4(x^2 - 2x + 1)]$$
$$= 3[-3x^2 + 8x - 4]$$
$$= -3(3x^2 - 8x + 4)$$

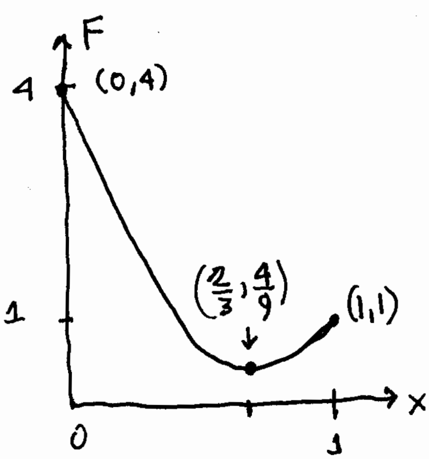
$$x = \frac{8 \pm \sqrt{64 - 16(3)}}{2\cdot 3} = \frac{4 \pm \sqrt{16-12}}{3} = \frac{4 \pm 2}{3}$$

$= \cancel{2}, \frac{2}{3}$ in interval $0 \le x \le 1$ (one critical pt in interval)

endpoints
$f(0) = 0 + 4(1)^3 = 4$
$f(1) = 1 + 4(0)^3 = 1$

crit: $f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 + 4\left(1 - \frac{2}{3}\right)^3 = \frac{8}{27} + 4\left(\frac{1}{27}\right)$

$= \frac{12}{27} = \frac{4}{9} \quad \to$ compare values



a) Max: $x = 0 \to y = 1 \to F = 4$ so:
This combination of the two numbers is maximum when the first number is 0 and the second number is 1.

b) Min: $x = 2/3 \to y = 1/3 \to F = 4/9$ so:
This combination of the two numbers is minimum when the first number is 2/3 and the second number is 1/3.

Since the roadmap problem was somewhat time consuming, it moved to the end.