MAT2705-01/04 03S Quiz 9 Print Name (Last, First) ______ last ______ first ______ /10

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (not decimal approximations, if possible).

$$\underline{x}' = \underline{A}\,\underline{x}\,,\quad \underline{x}(0) = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{A} = \frac{1}{20}\begin{bmatrix} -10 & 0 & 0 \\ 10 & -5 & 0 \\ 0 & 5 & -4 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{bmatrix}$$

$$\lambda = -\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{5} \qquad \underline{B} = \begin{bmatrix} 3 & 0 & 0 \\ -6 & 1 & 0 \\ 5 & -5 & 1 \end{bmatrix} \qquad \underline{B}^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2 & 1 & 0 \\ 25/3 & 5 & 1 \end{bmatrix} \qquad \underline{x} = \underline{B}\,\underline{y}$$

a) Evaluate $A_B = \underline{B}^{-1}\underline{A}\,\underline{B}$

b) Write out the system of DEs as 3 equations instead of the single matrix equation.

c) Write out the new system of DEs satisfied by y_1, y_2, y_3.

d) What is the general solution of these latter equations?

e) What is the general solution of the original equations?

f) Find the solution which satisfies the initial conditions. [Give result $x_1 = \cdots, x_2 = \cdots$, etc]

g) How long does it take for x_1 to decay to 1% of its initial value?

a) $$B^{-1}AB = \begin{bmatrix} 1/3 & 0 & 0 \\ 2 & 1 & 0 \\ 25/3 & 5 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ -6 & 1 & 0 \\ 5 & -5 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} -3/2 & 0 & 0 \\ 3/2+3/2 & -1/4 & 0 \\ -3/2-1 & 1/4+1 & -1/5 \end{bmatrix} = \begin{bmatrix} -3/2 & 0 & 0 \\ 3 & -1/4 & 0 \\ -5/2 & 5/4 & -1/5 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ -3+3 & -1/4 & 0 \\ -25/2+15-5/2 & -5/4+5/4 & -1/5 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & -1/5 \end{bmatrix}$$

b) $$\frac{dx_1}{dt} = -\tfrac{1}{2}x_1$$
$$\frac{dx_2}{dt} = \tfrac{1}{2}x_1 - \tfrac{1}{4}x_2$$
$$\frac{dx_3}{dt} = \tfrac{1}{4}x_2 - \tfrac{1}{5}x_3$$

c) d)
$$\frac{dy_1}{dt} = -\tfrac{1}{2}y_1 \rightarrow y_1 = c_1 e^{-\frac{1}{2}t}$$
$$\frac{dy_2}{dt} = -\tfrac{1}{4}y_2 \rightarrow y_2 = c_2 e^{-\frac{1}{4}t}$$
$$\frac{dy_3}{dt} = -\tfrac{1}{5}y_2 \rightarrow y_3 = c_3 e^{-\frac{1}{5}t}$$

e) $$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{B}\,\underline{y} = \begin{bmatrix} 3 & 0 & 0 \\ -6 & 1 & 0 \\ 5 & -5 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{-t/2} \\ c_2 e^{-t/4} \\ c_3 e^{-t/5} \end{bmatrix} = \begin{bmatrix} 3c_1 e^{-t/2} \\ -6c_1 e^{-t/2} + c_2 e^{-t/4} \\ 5c_1 e^{-t/2} - 5c_2 e^{-t/4} + c_3 e^{-t/5} \end{bmatrix}$$

f) $$\begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -6 & 1 & 0 \\ 5 & -5 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2 & 1 & 0 \\ 25/3 & 5 & 1 \end{bmatrix}\begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 30 \\ 125 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= 15e^{-t/2} \\ x_2 &= -30e^{-t/2} + 30e^{-t/4} \\ x_3 &= 25e^{-t/2} - 150e^{-t/4} + 125e^{-t/5} \end{aligned}}$$

g) $$x_1 = 15e^{-t/2} = .01(15)$$
$$e^{-t/2} = .01$$
$$-t/2 = \ln .01$$
$$t = -2\ln .01 = -2\ln 10^{-2} = 4\ln 10$$
$$\approx \boxed{9.21}$$