MAT 2705 035 FINAL EXAM ANSWERS

① a) $2y''+16y'+50y=52\sin 3t$

$\rightarrow y''+8y'+25y=26\sin 3t$ (standard form)

$r^2+8r+25=0$

$r=-\frac{8\pm\sqrt{64-100}}{2}=-\frac{8\pm 6i}{2}=-4\pm 3i$

$y_h = e^{-4t}(c_1\cos 3t+c_2\sin 3t)$

$\sin 3t$ not soln of hom eq so:

$25\,[\,y_p = c_3\cos 3t+c_4\sin 3t\,]$

$8\,[\,y_p' = -3c_3\sin 3t+3c_4\cos 3t\,]$

$\underline{1\,[\,y_p''=-9c_3\cos 3t-9c_4\sin 3t\,]}$

$y_p''+8y_p'+25y_p = [(25-9)c_3+24c_4]\cos 3t + [-24c_3+(25-9)c_4]\sin 3t$

$=\underbrace{(16c_3+24c_4)}_{0}\cos 3t+\underbrace{(-24c_3+16c_4)}_{26}\sin 3t = 26\sin 3t$

$16c_3+24c_4=0$
$-24c_3+16c_4=26$

$\begin{bmatrix}16&24&0\\-24&16&26\end{bmatrix}\rightarrow\begin{bmatrix}1&3/2&0\\-12&8&13\end{bmatrix}$ ($+12$ $+18$ $+0$)

$\rightarrow\begin{bmatrix}1&3/2&0\\0&26&13\end{bmatrix}\rightarrow\begin{bmatrix}1&3/2&0\\0&1&1/2\end{bmatrix}\rightarrow\begin{bmatrix}1&0&-3/4\\0&1&1/2\end{bmatrix}$ $c_1=-3/4$, $c_2=1/2$

$y_p=-\frac34\cos 3t+\frac12\sin 3t$

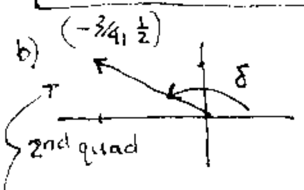
$y=y_h+y_p=e^{-4t}(c_1\cos 3t+c_2\sin 3t)+-\frac34\cos 3t+\frac12\sin 3t$

$y'=-4e^{-4t}(c_1\cos 3t+c_2\sin 3t)+e^{-4t}(-3c_1\sin 3t+3c_2\cos 3t)+\frac94\sin 3t+\frac32\cos 3t$

$y(0)=c_1-3/4=0$ $\quad c_1=3/4$

$y'(0)=-4c_1+3c_2+3/2=0$ $\quad c_2=-\frac32+\frac43\left(\frac34\right)=\frac12$

$$\boxed{y=\left(\frac34\cos 3t+\frac12\sin 3t\right)e^{-4t}+\left(-\frac34\cos 3t+\frac12\sin 3t\right)}$$

b) 

$y_p=-\frac34\cos 3t+\frac12\sin 3t$

$A=\sqrt{\left(\frac34\right)^2+\left(\frac24\right)^2}=\frac14\sqrt{13}\approx$

$\tan\delta=\frac{1/2}{-3/4}=-2/3$

$\delta=\pi-\arctan 2/3\approx\pi-0.5880\approx 2.5536$ $(\approx 146°)$

$$\boxed{y_p=\frac{\sqrt{13}}{4}\cos(3t-2.5536)}$$

② a) $\det(A-\lambda I)=\det\begin{bmatrix}2-\lambda&1&1\\2&3-\lambda&2\\1&1&2-\lambda\end{bmatrix}\overset{\text{technology}}{=}-\lambda^3+7\lambda^2-11\lambda+5=(\lambda+1)^2(\lambda+5)$

$\rightarrow\lambda=1,1,5$

b) $\lambda=1$: $A-\lambda I=\begin{bmatrix}1&1&1\\2&2&2\\1&1&1\end{bmatrix}\rightarrow\overset{\text{LFF}}{\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$x_1+x_2+x_3=0$
$0=0$
$0=0$

$x_2=t_1$
$x_3=t_2$
$x_1=-t_1-t_2$

② b) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-t_1-t_2\\t_1\\t_2\end{bmatrix}=t_1\underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{b_1}+t_2\underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{b_2}$

$\lambda=5$: $A-\lambda I=\begin{bmatrix}-3&1&1\\2&-2&2\\1&1&-3\end{bmatrix}\rightarrow\begin{bmatrix}1&-1&1\\-3&1&1\\1&1&-3\end{bmatrix}\rightarrow\begin{bmatrix}1&-1&1\\0&-2&4\\0&2&-4\end{bmatrix}$

$\rightarrow\begin{bmatrix}1&-1&1\\0&1&-2\\0&0&0\end{bmatrix}\rightarrow\underset{\text{LLF}}{\begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$x_1-x_3=0$ $\quad x_1=t$
$x_2-2x_3=0$ $\quad x_2=2t$
$0=0$

$x_3=t$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}t\\2t\\t\end{bmatrix}=t\underbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}}_{b_3}$

$$\boxed{\begin{aligned}\vec b_1&=[-1,1,0]\\\vec b_2&=[-1,0,1]\\\vec b_3&=[1,2,1]\end{aligned}}$$

c) $\underline{B}=\begin{bmatrix}-1&-1&1\\1&0&2\\0&1&1\end{bmatrix}$ $\quad B^{-1}AB=\begin{bmatrix}1&0&0\\0&1&0\\0&0&5\end{bmatrix}$ technology or by hand

$\underline{B^{-1}}=\frac14\begin{bmatrix}-2&2&-2\\-1&-1&3\\1&1&1\end{bmatrix}$ technology

d) $\underline{x}=c_1e^{\lambda_1t}\underline{b_1}+c_2e^{\lambda_2t}\underline{b_2}+c_3e^{\lambda_3t}\underline{b_3}$

$=c_1e^t\begin{bmatrix}-1\\1\\0\end{bmatrix}+c_2e^t\begin{bmatrix}-1\\0\\1\end{bmatrix}+c_3e^{5t}\begin{bmatrix}1\\2\\1\end{bmatrix}$

$=B\begin{bmatrix}c_1e^t\\c_2e^t\\c_3e^{5t}\end{bmatrix}=\begin{bmatrix}-c_1e^t-c_2e^t+c_3e^{5t}\\c_1e^t+2c_3e^{5t}\\c_2e^t+c_3e^{5t}\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$$\boxed{\begin{aligned}x_1&=-c_1e^t-c_2e^t+c_3e^{5t}\\x_2&=c_1e^t+2c_3e^{5t}\\x_3&=c_2e^t+c_3e^{5t}\end{aligned}}$$ gen soln

e) $x_1(0)=-c_1-c_2+c_3=1$
$x_2(0)=c_1+2c_3=1$
$x_3(0)=c_2+c_3=2$

$\begin{bmatrix}-1&-1&1&1\\1&0&2&1\\0&1&1&2\end{bmatrix}\rightarrow\begin{bmatrix}1&0&2&1\\-1&-1&1&1\\0&1&1&2\end{bmatrix}\rightarrow\begin{bmatrix}1&0&2&1\\0&-1&3&2\\0&1&1&2\end{bmatrix}$

$\rightarrow\begin{bmatrix}1&0&2&1\\0&0&4&4\\0&1&1&2\end{bmatrix}\rightarrow\begin{bmatrix}1&0&2&1\\0&1&1&2\\0&0&1&1\end{bmatrix}\rightarrow\begin{bmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&1\end{bmatrix}$ $\quad c_1=-1$, $c_2=1$, $c_3=1$

$$\boxed{\begin{aligned}x_1&=e^t-e^t+e^{5t}=e^{5t}\\x_2&=-e^t+2e^{5t}\\x_3&=e^t+e^{5t}\end{aligned}}$$ IVP soln

(2) f) $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1+x_2+x_3 \\ 2x_1+3x_2+2x_3 \\ x_1+x_2+2x_3 \end{bmatrix}$

$x_1 = e^{5t}$ $\quad x_1' = 5e^{5t}$

$x_2 = -e^t + 2e^{5t}$ $\quad x_2' = -e^t + 10e^{5t}$

$x_3 = e^t + e^{5t}$ $\quad x_3' = e^t + 5e^{5t}$

$$\boxed{\begin{array}{ll} x_1' = 2x_1+x_2+x_3 & g)\ x_1(0)=1 \\ x_2' = 2x_1+3x_2+2x_3 & x_2(0)=1 \\ x_3' = x_1+x_2+2x_3 & x_3(0)=2 \end{array}}$$

LHS = RHS

$\rightarrow 5e^{5t} = 2(e^{5t}) + (-e^t+2e^{5t}) + (e^t+e^{5t}) = 5e^{5t}$ ✓

$\rightarrow -e^t+10e^{5t} = 2(e^{5t}) + 3(-e^t+2e^{5t}) + 2(e^t+e^{5t}) = -e^t+10e^{5t}$ ✓

$\rightarrow e^t+5e^{5t} = (e^{5t}) + (-e^t+2e^{5t}) + 2(e^t+e^{5t}) = e^t+5e^{5t}$ ✓

g) $x_1(0) = e^0 = 1$ ✓

$x_2(0) = -e^0 + 2e^0 = 1$ ✓

$x_3(0) = e^0 + e^0 = 2$ ✓

(3) $A = \begin{bmatrix} -4 & -3 \\ 3 & -4 \end{bmatrix}$ $\quad \det(A-\lambda I) = \det\begin{bmatrix} -4-\lambda & -3 \\ 3 & -4-\lambda \end{bmatrix} = (\lambda+4)^2 + 9 = \lambda^2+8\lambda+25 = 0$

$\lambda = \dfrac{-8 \pm \sqrt{64-100}}{2} = -4 \pm 3i$

$\lambda = -4+3i$

$A-\lambda I = \begin{bmatrix} -4-(4+3i) & -3 \\ 3 & -4-(4+3i) \end{bmatrix} = \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \rightarrow \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}}_{L\ F}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 - ix_2 = 0$, $0=0$

$x_2 = t \quad x_1 = it$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}_{\vec{b}_1}$ $\qquad \vec{b}_2 = \overline{\vec{b}_1} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$B = (b_1\ b_2) = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

$B^{-1}AB = \begin{pmatrix} -4+3i & 0 \\ 0 & -4-3i \end{pmatrix}$

$\underline{X} = \underbrace{c_1 e^{(-4+3i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}} + c_2 e^{(-4-3i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

$= e^{-4t}(\cos 3t + i\sin 3t)\begin{bmatrix} i \\ 1 \end{bmatrix} = e^{-4t}\begin{bmatrix} -\sin 3t + i\cos 3t \\ \cos 3t + i\sin 3t \end{bmatrix}$

$= \begin{bmatrix} -e^{-4t}\sin 3t \\ e^{-4t}\cos 3t \end{bmatrix} + i\begin{bmatrix} e^{-4t}\cos 3t \\ e^{-4t}\sin 3t \end{bmatrix}$

$\underline{X} = a\begin{bmatrix} -e^{-4t}\sin 3t \\ e^{-4t}\cos 3t \end{bmatrix} + b\begin{bmatrix} e^{-4t}\cos 3t \\ e^{-4t}\sin 3t \end{bmatrix} = \begin{pmatrix} e^{-4t}(-a\sin 3t + b\cos 3t) \\ e^{-4t}(a\cos 3t + b\sin 3t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad b = 1$, $a = -1$

$$\boxed{\begin{array}{l} x_1 = e^{-4t}(\sin 3t + \cos 3t) \\ x_2 = e^{-4t}(-\cos 3t + \sin 3t) \end{array}}$$