MAT2705-03S TEST 3 TAKEHOME answers

① $(D^3+3D^2+4D+12)y = f(t)$

a) $r^3+3r^2+4r+12=0$

$(r+3)(r^2+4)=0$

$r=-3\,,\ \pm 2i$

b) $\boxed{y = c_1e^{-3t}+c_2\cos 2t + c_3 \sin 2t}$

c) $y' = -3c_1e^{-3t} - 2c_2\sin 2t + 2c_3\cos 2t$

$y'' = 9c_1e^{-3t} - 4c_2\cos 2t - 4c_3 \sin 2t$

$$\begin{aligned} y(0) &= c_1 + c_2 &= 13\\ y'(0) &= -3c_1 + 2c_3 &= 0\\ y''(0) &= 9c_1 - 4c_2 &= 0 \end{aligned}$$

$$\begin{bmatrix}1&1&0&13\\-3&0&2&0\\9&-4&0&0\end{bmatrix} \xrightarrow{\text{rref}^*} \begin{bmatrix}1&0&0&4\\0&1&0&9\\0&0&1&6\end{bmatrix} \quad \begin{aligned}c_1&=4\\c_2&=9\\c_3&=6\end{aligned}$$

$$\boxed{y = 4e^{-3t}+9\cos 2t + 6\sin 2t}$$

$$* \begin{bmatrix}1&1&0&13\\-3_{+3}&0_{+3}&2_{+0}&0_{+39}\\9_{-9}&-4_{-9}&0_{-0}&0_{-117}\end{bmatrix} \to \begin{bmatrix}1&1&0&13\\0&3&2&39\\0&-13&0&-13\cdot 9\end{bmatrix} \to$$

$$\begin{bmatrix}1&1&0&13\\0&3&2&39\\0&1&0&9\end{bmatrix} \to \begin{bmatrix}1&1_{-1}&0_{-0}&13_{-9}\\0&1&0&9\\0&3_{-3}&2_{0}&39_{-27}\end{bmatrix} \to \begin{bmatrix}1&0&0&4\\0&1&0&9\\0&0&2&12\end{bmatrix} \to$$

$$\begin{bmatrix}1&0&0&4\\0&1&0&9\\0&0&1&6\end{bmatrix}$$

check by backsub

$$\begin{aligned} 12[\,y &= 4e^{-3t} + 9\cos 2t + 6\sin 2t\,]\\ 4[\,y' &= -12e^{-3t} - 18\sin 2t + 12\cos 2t\,]\\ 3[\,y'' &= 36e^{-3t} - 36\cos 2t - 24\sin 2t\,]\\ 1[\,y''' &= -108e^{-3t} + 72\sin 2t - 48\cos 2t\,] \end{aligned}$$

$$\text{LHS} = \begin{pmatrix}-108\\+108\\-48\\+48\end{pmatrix}e^{-3t} + \begin{pmatrix}72\\-72\\-72\\+72\end{pmatrix}\sin 2t + \begin{pmatrix}-48\\-108\\+48\\+108\end{pmatrix}\cos 2t = 0 \ \checkmark$$

oops, read more carefully bob. check matrix soln:

$$\begin{bmatrix}1&1&0\\-3&0&2\\9&-4&0\end{bmatrix}\begin{bmatrix}4\\9\\6\end{bmatrix} = \begin{bmatrix}4+9\\-12+12\\36-36\end{bmatrix} = \begin{bmatrix}13\\0\\0\end{bmatrix} \checkmark$$

d) $y_{\text{sinusoid}} = 9\cos 2t + 6\sin 2t = 3(3\cos 2t + 2\sin 2t) = A\cos(2t-\delta)$

$A = 3\sqrt{3^2+2^2} = \boxed{3\sqrt{13} \approx 10.82}$

$\tan\delta = 2/3$ first quad $\to$ $\boxed{\delta = \arctan 2/3 \approx 0.5880 \approx 16.84^\circ}$

$\sqrt{13}$ 6 9

e) $y_{exp} = 4e^{-3t} = .01\,(10.82)$

$e^{-3t} = .01\,(10.82)/4$

$-3t = \ln(.01\,(10.82)/4) \quad t = -\frac{1}{3}\ln(.01\,(10.82)/4)$

$\approx \boxed{1.20}$

f)

```
> plot([4*exp(-3*t)+6*sin(2*t)+9*cos(2*t),4*exp(-3*t),6*sin(2*t)+9*c
  os(2*t)],t=0..2*Pi,color=[red,green,blue]);
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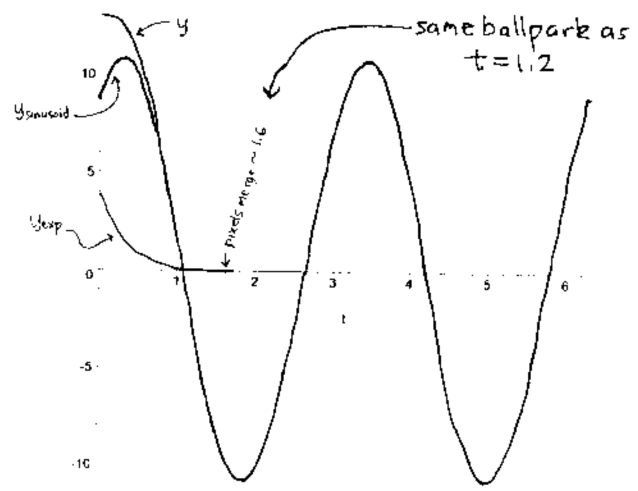

g) from b) basis of hom soln: $\{e^{-3t}, \cos 2t, \sin 2t\}$

so $f(t) = 30\cos t$ does not satisfy hom eqn, so

$$\begin{aligned} 12[\,y_p &= c_4\cos t + c_5 \sin t\,]\\ 4[\,y_p' &= -c_4\sin t + c_5\cos t\,]\\ 3[\,y_p'' &= -c_4\cos t - c_5\sin t\,]\\ 1[\,y_p''' &= c_4\sin t - c_5\cos t\,] \end{aligned}$$

$\text{LHS}_p = (-c_5 - 3c_4 + 4c_5 + 12c_4)\cos t + (c_4 - 3c_5 - 4c_4 + 12c_5)\sin t$

$= (3c_5 + 9c_4)\cos t + (-3c_4 + 9c_5)\sin t \overset{DE}{=} 30\cos t$

$$\begin{aligned} 9c_4 + 3c_5 &= 30\\ -3c_4 + 9c_5 &= 0 \end{aligned} \quad \begin{bmatrix}9&3&30\\-3&9&0\end{bmatrix} \to \begin{bmatrix}3&1&10\\-1&3&0\end{bmatrix} \to \begin{bmatrix}1&-3&0\\3&1&10\end{bmatrix}$$

$$\to \begin{bmatrix}1&-3&0\\0&10&10\end{bmatrix} \to \begin{bmatrix}1&-3&0\\0&1&1\end{bmatrix} \to \begin{bmatrix}1&0&3\\0&1&1\end{bmatrix} \quad \begin{aligned}c_4&=3\\c_5&=1\end{aligned}$$

$$\begin{aligned} 12[\,y_p &= 3\cos t + \sin t\,]\\ 4[\,y_p' &= -3\sin t + \cos t\,]\\ 3[\,y_p'' &= -3\cos t - \sin t\,]\\ 1[\,y_p''' &= 3\sin t - \cos t\,] \end{aligned}$$

$y_p''' + 3y_p'' + 4y_p' + 12y_p = (-1-9+4+36)\cos t = 30\cos t \checkmark$

$+(3-3-12+12)\sin t$

② $x'' + 10x' + 650x = 100\cos\omega t$

a) $1 \quad k_0 = 1/\tau_0 \quad \omega_0^2$

$\boxed{\tau_0 = \frac{1}{10},\ \omega_0 = \sqrt{650} \approx 25.50}$

$\boxed{Q = \omega_0\tau_0 = \frac{\sqrt{650}}{10} = \frac{\sqrt{26}}{2} \approx 2.55}$

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(2) b) $x'' + 10x' + 650x = 100\cos\omega t$

$(D^2+10D+650)x = 100\cos\omega t$

$r^2+10r+650=0$

$r = \dfrac{-10 \pm \sqrt{100-4\cdot 650}}{2} = \dfrac{-10 \pm i\sqrt{2500}}{2}$

$= \dfrac{-10 \pm 50i}{2} = -5 \pm 25i$

$\boxed{x_h = e^{-5t}(c_1\cos 25t + c_2 \sin 25t)}$

c) $650\,[x_p = c_3\cos 25t + c_4 \sin 25t]$

$10\,[x_p' = -25c_3 \sin 25t + 25c_4\cos 25t]$

$1\,[x_p'' = -625c_3\cos 25t - 625c_4 \sin 25t]$

$x_p''+10x_p'+650x_p = (25c_3+250c_4)\cos 25t = 100\cos 25t$
$\quad + (-250c_3+25c_4)\sin 25t$

$25c_3+250c_4 = 100$
$-250c_3+25c_4 = 0$

$\begin{bmatrix} 25 & 250 & 100 \\ -250 & 25 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 4 \\ -10 & 1 & 0 \end{bmatrix} \to$ (10 100 40)

$\begin{bmatrix} 1 & 10 & 4 \\ 0 & 101 & 40 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 4 \\ 0 & 1 & \frac{40}{101} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & (404-400)/101 \\ 0 & 1 & \frac{40}{101} \end{bmatrix}$

$c_3 = 4/101 \quad c_4 = 40/101$

$\boxed{x_p = (4\cos 25t + 40\sin 25t)/101} = A\cos(25t-\delta)$

$\frac{4}{101}(1,10)$ $\quad \boxed{A = \frac{4}{101}\sqrt{101} = \frac{4}{\sqrt{101}} \approx 0.3980}$

$\tan\delta = 10$ $\quad \boxed{\delta = \arctan 10 \approx 1.471}$ $\approx 84.3°$

d) $650\,[x_p = c_3\cos\omega t + c_4\sin\omega t]$

$10\,[x_p' = -c_3\omega\sin\omega t + c_4\omega\cos\omega t]$

$1\,[x_p'' = -c_3\omega^2\cos\omega t - c_4\omega^2\sin\omega t]$

$x_p''+10x_p'+650x_p = [(650-\omega^2)c_3 + 10\omega c_4]\cos\omega t = 100\cos\omega t$
$\quad + [-10\omega c_3 + (650-\omega^2)c_4]\sin\omega t$

$(650-\omega^2)c_3 + 10\omega c_4 = 100$
$-10\omega c_3 + (650-\omega^2)c_4 = 0$

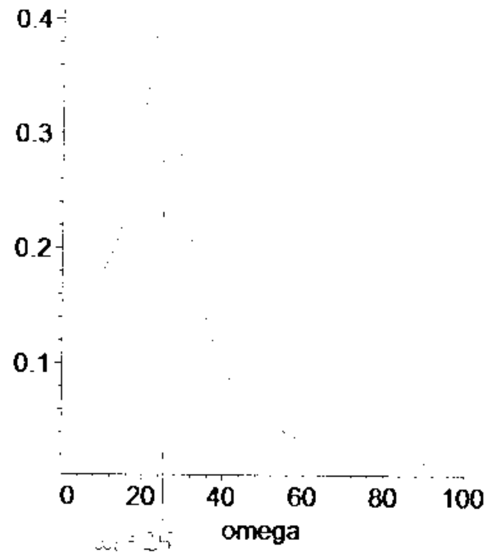
$\begin{bmatrix} (650-\omega^2) & 10\omega & 100 \\ -10\omega & (650-\omega^2) & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{10\omega}(650-\omega^2) & 0 \\ 650-\omega^2 & 10\omega & 100 \end{bmatrix} \to$

$\begin{bmatrix} 1 & -\frac{1}{10\omega}(650-\omega^2) & 0 \\ 0 & 10\omega + \frac{1}{10\omega}(650-\omega^2)^2 & 100 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{10\omega}(650-\omega^2) & 0 \\ 0 & 1 & \frac{100\cdot(10\omega)}{(650-\omega^2)^2+100\omega^2} \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 100(650-\omega^2)/D \\ 0 & 1 & 100(10\omega)/D \end{bmatrix}$ $\quad (c_3, c_4) = \dfrac{100\,(650-\omega^2,\ 10\omega)}{(650-\omega^2)^2+100\omega^2}$

$A(\omega) = \dfrac{100}{\sqrt{(650-\omega^2)^2+100\omega^2}}$

e)



f) $0 = A'(\omega) = 100(-\tfrac{1}{2})\left((650-\omega^2)^2+100\omega^2\right)^{-3/2}\left[2(650-\omega^2)(-2\omega) + 200\omega\right]$

$= -100\,D^{-3/2}\,\omega\,[100 - 2(650-\omega^2)]$

$= -100\,D^{-3/2}\,2\omega\,(\omega^2-600) \to \boxed{\omega = \sqrt{600} = 10\sqrt{6} \approx 24.49}$

very close but slightly less than $\omega_0 = 25$

$A(0) = 100/650$

$A(10\sqrt{6}) = 100/\sqrt{(650-600)^2 + 600\cdot 100}$

$\dfrac{A(10\sqrt{6})}{A(0)} = \dfrac{650}{\sqrt{50^2+60000}} \quad \dfrac{650}{250} = 2.6$ (recall $Q \approx 2.55$)

(3) a) $A-\lambda I = \begin{bmatrix} 11-\lambda & -6 & -2 \\ 20 & -11-\lambda & -4 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$\det(A-\lambda I) = -\lambda^3+\lambda^2+\lambda-1 = -(\lambda+1)(\lambda-1)^2 = 0$

$\boxed{\lambda = -1,\ 1 \quad \text{mult: } 1,\ 2}$ eigenvalues

b) $\lambda = -1$, $A-\lambda I = \begin{bmatrix} 12 & -6 & -2 \\ 20 & -10 & -4 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 12 & -6 & 0 \\ 20 & -10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to$

$\begin{bmatrix} 1 & -\frac12 & 0 \\ 1 & -\frac12 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{array}{ccc} L & F & L \\ \end{array} \begin{bmatrix} 1 & -\frac12 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_1 - \frac12 x_2 = 0$, $x_3 = 0$

$x_2 = t \to x_1 = \frac12 t$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t/2 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\vec{b}_1 = [1,2,0]}$ double to clear fractions.

$\lambda = 1$, $A-\lambda I = \begin{bmatrix} 10 & -6 & -2 \\ 20 & -12 & -4 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{array}{ccc} L & F & F \\ \end{array} \begin{bmatrix} 1 & -\frac35 & -\frac15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - \frac35 x_2 - \frac15 x_3 = 0$, $0=0$, $0=0$ $\quad x_1 = 3t_1/5 + t_2/5$, $x_2 = t_1$, $x_3 = t_2$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1\begin{bmatrix} 3/5 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\vec{b}_2 = [3,5,0],\ \vec{b}_3 = [1,0,5]}$ again doubling

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③ c) $\underline{B} = \text{augment}(\vec{b}_1, \vec{b}_2, \vec{b}_3)$

$$= \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \underline{B}^{-1} = \begin{bmatrix} -5 & 3 & 1 \\ 2 & -1 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}$$

d)
$$AB = \begin{bmatrix} 11 & -6 & -2 \\ 20 & -11 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 11-12 & 33-30 & 11-10 \\ 20-22 & 60-55 & 20-20 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 1 \\ -2 & +5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B^{-1}AB = \begin{bmatrix} -5 & 3 & 1 \\ 2 & -1 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 1 \\ -2 & +5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5-6 & -15+15 & -5+5 \\ -2+2 & 6-5 & 2-2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \checkmark$$

e) for $\lambda = 1$ $\quad \vec{b}_2 = [3,5,0]$, $\vec{b}_3 = [1,0,5]$

maple: $\quad [0,1,-3]$ not, $\quad [1,0,5]$ same

to get 0 in first component: $\quad [3,5,0] - 3[1,0,5] = [0,5,-15]$

divide by 5: $\quad \rightarrow [0,1,-3] = \frac{1}{5}(\vec{b}_2 - 3\vec{b}_3)$

so just a linear combination of the two we found.

f) coord trans: $\quad \underline{x} = \underline{B}\,\underline{y} \rightarrow \underline{y} = \underline{B}^{-1}\underline{x}$

$$\underline{x} = [1,2,5] \rightarrow \underline{y} = B^{-1}[1,2,5] = \boxed{[6,-2,1]}$$

check:
$$6\vec{b}_1 - 2\vec{b}_2 + \vec{b}_3 = \begin{matrix} 6[1,2,0] \\ -2[3,5,0] \\ +1[1,0,5] \end{matrix} = [6-6+1, 12-10+0, 0-0+5] = [1,2,5] \checkmark$$

④ $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = (2-\lambda)(4-\lambda) + 2$$
$$= \lambda^2 - 6\lambda + 8 + 2 = \lambda^2 - 6\lambda + 10$$

$$\lambda = \frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm i$$

$$\lambda = 3+i, \quad A - \lambda I = \begin{bmatrix} 2-3-i & 1 \\ -2 & 1-i \end{bmatrix} \rightarrow \begin{array}{cc} L & F \\ \left[\begin{matrix} 1 & -\frac{1}{2}(1-i) \\ 0 & 0 \end{matrix}\right. & \left.\right] \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - \tfrac{1}{2}(1-i)x_2 = 0 \qquad x_1 = \tfrac{1}{2}(1-i)t \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1-i)t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} \frac{1}{2}(1-i) \\ 1 \end{bmatrix}}_{\underline{b}_1}$$
$$0 = 0 \qquad x_2 = t$$

$$\lambda = 3-i \qquad \underline{b}_2 = \overline{\underline{b}_1} = \begin{bmatrix} \frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

④ continued)

$$\vec{b}_1 = \left[\tfrac{1}{2}(1-i), 1\right] \qquad \vec{b}_2 = \left[\tfrac{1}{2}(1+i), 1\right]$$

MAPLE: (half the time) $\quad [1, 1+i] \qquad [1, 1-i]$

divide thru by second entry to make 1:

$$\left[\frac{1}{1+i}, 1\right] \qquad \left[\frac{1}{1-i}, 1\right]$$

$$\frac{1}{(1+i)}\frac{(1-i)}{(1-i)} = \frac{1}{2}(1-i) \checkmark \qquad \xrightarrow{\text{c.c.}} \frac{1}{2}(1+i) \checkmark$$

half the time MAPLE gives our result directly. wierd but true.