MAT 2705-01/04 03S Test 1 answers

(1) $x^2\frac{dy}{dx}+(y+1)=0 \to \frac{dy}{dx}=-\frac{(y+1)}{x^2} \to \frac{dy}{dx}+\frac{1}{x^2}y=-\frac{1}{x^2}$

separable — linear standard form

a) $\int\frac{dy}{y+1}=-\int x^{-2}dx \to \ln|y+1| = -\left(\frac{x^{-1}}{-1}\right)+C_1 = \frac{1}{x}+C_1$

$|y+1| = e^{\frac{1}{x}+C_1} = e^{C_1}e^{1/x} \to y+1 = \underbrace{(\pm e^{C_1})}_{C_2}e^{1/x} = C_2 e^{1/x}$

$$\boxed{y=-1+C_2e^{1/x}}$$

b) $e^{-1/x}\left[\frac{dy}{dx}+\frac{1}{x^2}y=-\frac{1}{x^2}\right] \to \frac{d}{dx}\left(ye^{-1/x}\right) = -e^{-1/x}\frac{1}{x^2}$

$\int x^{-2}dx = \frac{x^{-1}}{-1} = -\frac{1}{x}$, $\quad e^{\int x^{-2}dx} = e^{\frac{x^{-1}}{-1}} = e^{-1/x}$

$ye^{-1/x} = -\int e^{-1/x}\underbrace{\frac{1}{x^2}dx}_{du} = -\int e^u du = -e^u + C_3 = C_3 - e^{-1/x}$

$u=-x^{-1}$
$du = -(-x^{-2})dx = x^{-2}dx$

$y = e^{1/x}(C_3 - e^{-1/x}) = C_3e^{1/x} - \underbrace{e^{1/x}e^{-1/x}}_{1} = -1 + C_3e^{1/x}$

$$\boxed{y=-1+C_3e^{1/x}}$$ same as above with $C_3=C_2$.

c) $y(1) = -1+C_3e^1 = 2 \to C_3e = 2+1 = 3 \to C_3 = 3/e$

$$\boxed{y=-1+\frac{3}{e}e^{1/x} = -1+3e^{1/x-1}}$$ (using rules of exponents in the second form)

d) $\frac{dy}{dx} = 0+\frac{3}{e}e^{1/x}\underbrace{(-x^{-2})}_{\frac{d}{dx}(x^{-1})} = -\frac{3}{e}\frac{e^{1/x}}{x^2}$

$x^2\frac{dy}{dx}+(y+1)=0 \to \cancel{x^2}\left(-\frac{3}{e}\frac{e^{1/x}}{\cancel{x^2}}\right)+\left(\cancel{-1}+\frac{3}{e}e^{1/x}\cancel{+1}\right)\overset{?}{=}0$

$-\frac{3}{e}e^{1/x}+\frac{3}{e}e^{1/x} = 0$

$0=0$ ✓

(2) a) $\frac{dP}{dt}\propto P^2 \to \boxed{\frac{dP}{dt}=kP^2 \quad P(0)=12,\ P(10)=24,\ P(t)=48\,?}$

$t=0 \leftrightarrow 1988$
$t=10 \leftrightarrow 1998$

b) $\int P^{-2}dP = \int k\,dt \to \frac{P^{-1}}{-1} = kt+C_1 \to P = \frac{-1}{C_1+kt}$ gen soln

$12 = P(0) = -\frac{1}{C_1+0} = -\frac{1}{C_1} \to C_1 = -\frac{1}{12} \to P = \frac{-1}{-\frac{1}{12}+kt} = \frac{12}{1-12kt}$

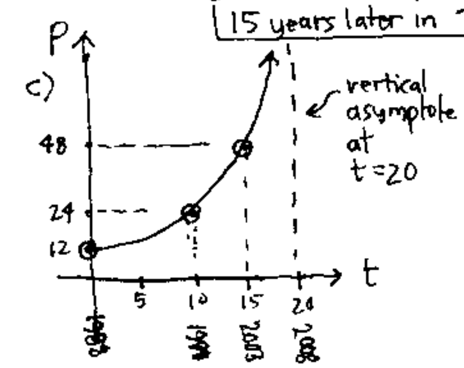
$24 = P(10) = \frac{12}{1-120k} \to 1-120k = \frac{12}{24} = \frac{1}{2} \to \frac{1}{2} = 120k \to k = \frac{1}{240} \to P = \frac{12}{1-\frac{12}{240}t} = \frac{12}{1-\frac{t}{20}}$

IVP soln $\boxed{P=\frac{240}{20-t}}$

$48 = P(t) = \frac{240}{20-t} \to 20-t = \frac{240}{48} = \frac{20}{4} = 5 \to t = 20-5 = 15 \leftrightarrow 2003$

$\boxed{\text{15 years later in 2003 there will be 48 alligators in the swamp.}}$

c)



d) P goes infinite as $t\to 20$, so the model must break down since a real population can only be finite.

$t=20 \leftrightarrow$ year 2008

e) $P = 240(20-t)^{-1}$ $\quad \frac{dP}{dt} = 240(-1)(20-t)^{-2}(-1) = \frac{240}{(20-t)^2}$

$\frac{dP}{dt} = \frac{1}{240}P^2 \to \frac{240}{(20-t)^2} \overset{?}{=} \frac{1}{240}\left(\frac{240}{20-t}\right)^2 = \frac{240}{(20-t)^2}$ ✓