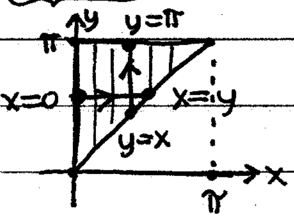
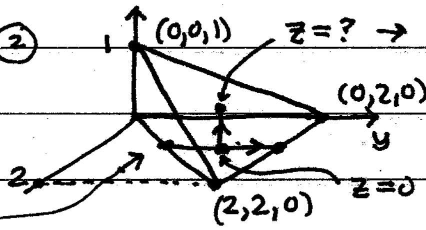
MAT2500-02/03 02S Test 3 (Takehome) Answers

① $\int_{x=0}^{x=\pi}\int_{y=x}^{y=\pi} \frac{\sin y}{y}\,dy\,dx = \boxed{\int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\,dx\,dy} = \int_0^{\pi}\left(\frac{\sin y}{y}\right)\underbrace{x\Big|_{x=0}^{x=y}}_{y}\,dy = \int_0^{\pi}\sin y\,dy$

$= -\cos y\Big|_0^{\pi} = -\cos\pi + \cos 0 = -(-1)+1 = \boxed{2}$



② 

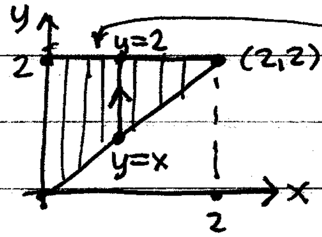
$z = ?$ → write eqn for plane → normal vector:

$(\langle 2,2,0\rangle - \langle 0,0,1\rangle)\times(\langle 0,2,0\rangle - \langle 0,0,1\rangle)$

$= \langle 2,2,-1\rangle \times \langle 0,2,-1\rangle$

$= \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 0 & 2 & -1 \end{vmatrix} = \langle -2+2,\ 0+2,\ 4-0\rangle$

$= \cancel{2}\langle 0,2,4\rangle = 2\underbrace{\langle 0,1,2\rangle}_{\vec{n}}$



$\vec{n}\cdot(\vec{r}-\vec{r_0}) = 0 \qquad \langle 0,1,2\rangle\cdot\langle x-0, y-0, z-1\rangle = 0$

$y + 2(z-1) = 0 \to y + 2z = 2 \to z = \frac{2-y}{2} = 1 - \frac{1}{2}y$ so:

$\iiint_E x^2+y^2\,dV = \boxed{\int_0^2\int_x^2\int_0^{1-y/2}(x^2+y^2)\,dz\,dy\,dx}$

$= \int_0^2\int_x^2 (x^2+y^2)\,z\Big|_{z=0}^{z=1-y/2}\,dy\,dx = \int_0^2\int_x^2 (x^2+y^2)(1-\frac{y}{2})\,dy\,dx = \int_0^2\int_x^2 x^2(1-\frac{y}{2}) + y^2 - \frac{y^3}{2}\,dy\,dx$

$= \int_0^2 x^2(y - \frac{y^2}{4}) + \frac{y^3}{3} - \frac{y^4}{8}\Big|_{y=x}^{y=2}\,dx = \int_0^2 x^2\underbrace{\left[2-\frac{2^2}{4}\right]}_{1} + \underbrace{\left(\frac{2^3}{3}-\frac{2^4}{8}\right)}_{\frac{8}{3}-2=\frac{2}{3}} - x^2(x-\frac{x^2}{4}) - \frac{x^3}{3} + \frac{x^4}{8}\,dx$ $\quad \frac{1}{4}+\frac{1}{8} = \frac{3}{8}$

$= \int_0^2 \frac{2}{3} + x^2 - \frac{4}{3}x^3 + \frac{3}{8}x^4\,dx$

$= \frac{2}{3}x + \frac{x^3}{3} - \frac{4x^4}{3\cdot 4} + \frac{3x^5}{8\cdot 5}\Big|_0^2 = \frac{4}{3} + \frac{8}{3} - \frac{16}{3} + \frac{4\cdot 3}{5} = \frac{12}{5} - \frac{4}{3} = \frac{36-20}{15} = \boxed{\frac{16}{15}}$

③ $\int_{x=0}^{2}\int_{y=-\sqrt{4-x^2}}^{0}\int_{z=2-\sqrt{4-x^2-y^2}}^{z=2+\sqrt{4-x^2-y^2}} z\,dz\,dy\,dx$

$dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$y^2 = 4-x^2 \to x^2+y^2 = 4$

$z - 2 = \pm\sqrt{4-x^2-y^2}$

$(z-2)^2 = 4-x^2-y^2$

$x^2+y^2+(z-2)^2 = 2^2$

Center $(0,0,2)$
radius 2

→ 1/4 of sphere cut vertically

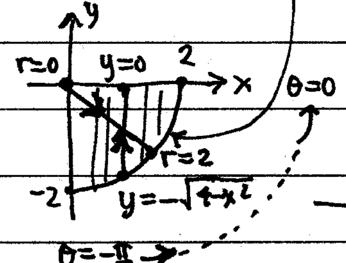
$\begin{cases} z = \rho\cos\varphi \\ r = \rho\sin\varphi \end{cases}$

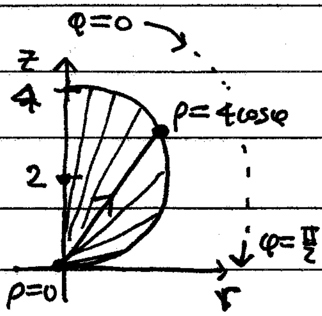
$r^2 + (z-2)^2 = 4$

$(\rho\sin\varphi)^2 + (\rho\cos\varphi - 2)^2 = 4$

$\underbrace{\rho^2\sin^2\varphi + \rho^2\cos^2\varphi}_{\rho^2} - 4\rho\cos\varphi + \cancel{4} = \cancel{4}$

$\rho^2 = 4\rho\cos\varphi \to \boxed{\rho = 4\cos\varphi}$





$= \boxed{\int_{-\frac{\pi}{2}}^{0}\int_0^{\frac{\pi}{2}}\int_0^{4\cos\varphi} \underbrace{(\rho\cos\varphi)(\rho^2\sin\varphi}_{\rho^3\cos\varphi\sin\varphi}\,d\rho\,d\varphi\,d\theta)} = \int_{-\frac{\pi}{2}}^{0}\int_0^{\frac{\pi}{2}} \underbrace{\frac{\rho^4}{4}\cos\varphi\sin\varphi\Big|_{\rho=0}^{\rho=4\cos\varphi}}_{\frac{4^4}{4}\cos^5\varphi\sin\varphi}\,d\varphi\,d\theta$

$= \frac{4^4}{4}\int_{-\frac{\pi}{2}}^{0}\int_0^{\frac{\pi}{2}} \underbrace{\cos^5\varphi}_{u^5}\,\underbrace{\sin\varphi\,d\varphi}_{-du}\,d\theta = -\frac{4^3}{6}\int_{-\frac{\pi}{2}}^{0}\cos^6\varphi\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}}\,d\theta = \left(\frac{\pi}{2}\right)\left[-\frac{4^3}{6}\left(\underbrace{\cos^6\frac{\pi}{2}}_{0} - \underbrace{\cos^6 0}_{1}\right)\right] = \frac{4^3}{6}\left(\frac{\pi}{2}\right) = \boxed{\frac{16\pi}{3}}$

$\int u^5(-du) = -\frac{u^6}{6} = -\frac{\cos^6\varphi}{6}$

$\theta\Big|_{-\pi/2}^{0} = 0 - (-\pi/2) = \pi/2$

Note center of gravity is at center of sphere by symmetry so

$\left.\bar{z} = \frac{\iiint z\,dV}{V} = 2\right\}$ so $\iiint_O z\,dV = 2V = 2\cdot\frac{4\pi}{3}\cdot 2^3 = 4\left(\frac{16\pi}{3}\right)$

but the integral is only over 1/4 the sphere, yielding $\frac{16\pi}{3}$

MAT2500-02/03 02S Test 3 (Takehome) Answers (2)

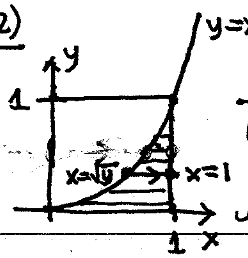
④ a) $\displaystyle\int_{y=0}^{y=1}\int_{x=\sqrt{y}}^{x=1}\int_{z=0}^{z=y} xy\,dz\,dx\,dy$

$xy\,z\Big|_{z=0}^{z=y} = xy^2$

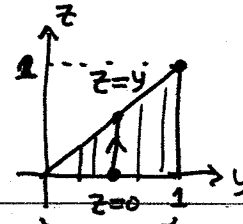
$\dfrac{x^2y^2}{2}\Big|_{x=\sqrt{y}}^{x=1} = \frac{1}{2}(1\cdot y^2 - y\cdot y^2) = \frac{1}{2}(y^2-y^3)$

$\frac{1}{2}\left(\frac{y^3}{3}-\frac{y^4}{4}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{2}\frac{(4-3)}{12} = \boxed{\frac{1}{24}}$

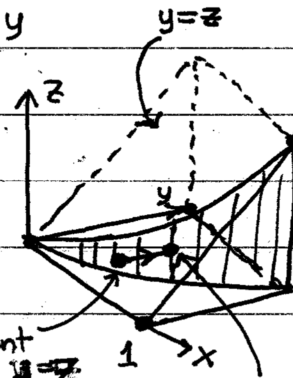
b)



x-y base region (outer double integral)



$0\le y\le 1$ (inner integral)



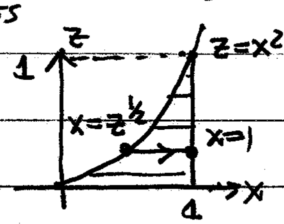
front plane $y=z$

back wall $y=x^2$

project region onto z-x plane

top edge: $y=z$ intersects $y=x^2 \to z=x^2$

so.:



$= \displaystyle\int_0^1\int_{z^{1/2}}^1\int_z^{x^2} xy\,dy\,dx\,dz$

$\dfrac{xy^2}{2}\Big|_{y=z}^{y=x^2} = \dfrac{x(x^2)^2-xz^2}{2} = \dfrac{x^5-xz^2}{2}$

$\frac{1}{2}\left(\frac{x^6}{6}-\frac{x^2}{2}z^2\right)\Big|_{x=z^{1/2}}^{x=1} = \frac{1}{2}\left(\frac{1}{6}-\frac{z^2}{2}-\frac{z^3}{6}+\frac{z^3}{2}\right)$

$= \displaystyle\int_0^1 \frac{1}{4}\left(\frac{1}{3}-z^2+\frac{2}{3}z^3\right)dz = \frac{1}{4}\left(\frac{z}{3}-\frac{z^3}{3}+\frac{2z^4}{3\cdot 4}\right)\Big|_0^1 = \frac{1}{4}\left(\frac{1}{3}-\frac{1}{3}+\frac{1}{6}\right) = \boxed{\frac{1}{24}}$

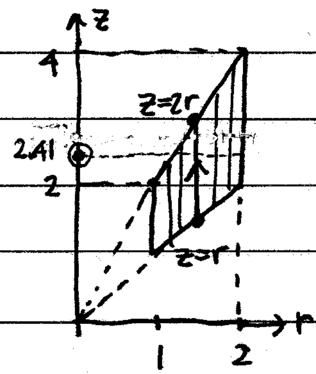
⑤ a)

$z=\sqrt{x^2+y^2}=r$

$z=2\sqrt{x^2+y^2}=2r$

$x^2+y^2=1\to r^2=1\to r=1$

$x^2+y^2=4\to r^2=4\to r=2$



$\langle\bar{x},\bar{y},\bar{z}\rangle = \dfrac{\left\langle \iiint_E x\,dV, \iiint_E y\,dV, \iiint_E z\,dV\right\rangle}{\iiint_E 1\,dV} \leftarrow V$

b) $V = \displaystyle\int_0^{2\pi}\int_1^2\int_r^{2r} 1\,(r\,dz\,dr\,d\theta) = 2\pi\int_1^2\int_r^{2r} r\,dz\,dr = 2\pi\int_1^2 rz\Big|_{z=r}^{z=2r} dr = 2\pi\int_1^2 \underbrace{r(2r-r)}_{r^2}\,dr$

$= 2\pi\,\dfrac{r^3}{3}\Big|_1^2 = \dfrac{2\pi}{3}(8-1) = \boxed{\dfrac{14\pi}{3}}$

$\displaystyle\iiint_E x\,dV = \int_0^{2\pi}\int_1^2\int_r^{2r}(r\cos\theta)(r\,dz\,dr\,d\theta) = \underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{+\sin\theta|_0^{2\pi}=0}\int_1^2\int_r^{2r} r^2\,dz\,dr$

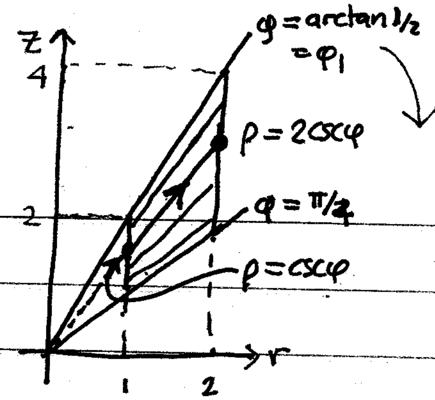
$\displaystyle\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_r^{2r}(r\sin\theta)(r\,dz\,dr\,d\theta) = \underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{-\cos\theta|_0^{2\pi}=0}\int_1^2\int_r^{2r} r^2\,dz\,dr$

$\boxed{\bar{x}=0=\bar{y}}$ on z-axis as it should from rotational symmetry

$\displaystyle\iiint_E z\,dV = \int_0^{2\pi}\int_1^2\int_r^{2r} z\,(r\,dz\,dr\,d\theta) = 2\pi\int_1^2 \frac{z^2}{2}r\Big|_{z=r}^{z=2r} dr = \pi\int_1^2(4-1)r^3\,dr = \frac{3\pi r^4}{4}\Big|_1^2 = \frac{3\pi}{4}(16-1) = 45\pi/4$

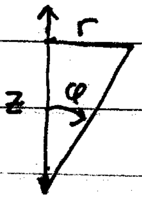
$\bar{z} = \dfrac{45\pi/4}{14\pi/3} = \dfrac{3}{4}\cdot\dfrac{45}{14} = \boxed{\dfrac{135}{56}} \approx 2.41$ (seems about right) see above diagram

MAT2500-02/03 025 Test 3 (Takehome) Answers (3)



c) $r=1 \to \rho\sin\varphi = 1 \to \rho = \csc\varphi$

$r=2 \to \rho\sin\varphi = 2 \to \rho = 2\csc\varphi$



$\tan\varphi = \frac{r}{z}$

$z = r \to \tan\varphi = 1,\ \varphi = \pi/4,\ \cot\varphi = 1$

$z = 2r \to \tan\varphi = \frac{1}{2},\ \varphi = \arctan 1/2,\ \cot\varphi = 1/(1/2) = 2$

d) $V = \int_0^{2\pi}\int_{\arctan 1/2}^{\pi/4}\int_{\csc\varphi}^{2\csc\varphi} 1\,(\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta) = 2\pi\int_{\arctan 1/2}^{\pi/4} \frac{\rho^3}{3}\sin\varphi\,\Big|_{\rho=\csc\varphi}^{\rho=2\csc\varphi} d\varphi$

$= \frac{2\pi}{3}\int_{\arctan 1/2}^{\pi/4}(8-1)\csc^3\varphi\sin\varphi\, d\varphi = \frac{14\pi}{3}\int_{\arctan 1/2}^{\pi/4}\csc^2\varphi\, d\varphi = \frac{14\pi}{3}(-\cot\varphi)\Big|_{\arctan 1/2}^{\pi/4}$

$= \frac{14\pi}{3}\left(-\underbrace{\cot\frac{\pi}{4}}_{1} + \underbrace{\cot(\arctan 1/2)}_{2}\right) = \boxed{\frac{14\pi}{3}}$

$\iiint x\, dV = \int_0^{2\pi}\int_{\varphi_1}^{\pi/4}\int_{\csc\varphi}^{2\csc\varphi}(\rho\sin\varphi\cos\theta)(\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta)$

$= \underbrace{\int_0^{2\pi}\cos\theta\, d\theta}_{\sin\theta|_0^{2\pi}=0}\int_{\varphi_1}^{\pi/4}\int_{\csc\varphi}^{2\csc\varphi}\rho^3\sin^2\varphi\, d\rho\, d\varphi = 0$

$\iiint y\, dV = \int_0^{2\pi}\int_{\varphi_1}^{\pi/4}\int_{\csc\varphi}^{2\csc\varphi}(\rho\sin\varphi\sin\theta)(\rho^2\sin\varphi\, d\varphi\, d\theta)$

$= \underbrace{\int_0^{2\pi}\sin\theta\, d\theta}_{-\cos\theta|_0^{2\pi}=0}\int_{\varphi_1}^{\pi/4}\int_{\csc\varphi}^{2\csc\varphi}\rho^3\sin^2\varphi\, d\rho\, d\varphi = 0$

$\boxed{\bar{x} = 0 = \bar{y}}$

$\iiint z\, dV = \int_0^{2\pi}\int_{\varphi_1}^{\pi/4}\int_{\csc\varphi}^{2\csc\varphi}(\rho\cos\varphi)(\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta)$

$= (2\pi)\int_{\varphi_1}^{\pi/4}\underbrace{\int_{\csc\varphi}^{2\csc\varphi}\rho^3\cos\varphi\sin\varphi\, d\rho}\, d\varphi$

$\frac{\rho^4}{4}\cos\varphi\sin\varphi\Big|_{\rho=\csc\varphi}^{\rho=2\csc\varphi} = \frac{16-1}{4}\csc^4\varphi\cos\varphi\sin\varphi = \frac{15}{4}\cot\varphi\csc^2\varphi$

$= \frac{15\pi}{2}\int_{\varphi_1}^{\pi/4}\underbrace{\cot\varphi}_{u}\ \underbrace{\csc^2\varphi\, d\varphi}_{-du} = \frac{15\pi}{4}\underbrace{(-\cot^2\varphi)\Big|_{\arctan 1/2}^{\pi/4}}_{-1+2^2=3} = \frac{45\pi}{4}$

$\int u(-du) = -\frac{u^2}{2} = -\frac{\cot^2\varphi}{2}$

$\bar{z} = \frac{45\pi/4}{14\pi/3} = \boxed{\frac{135}{56}}$