MAT1505 Q5 in class exercise with followup (~Q5)

a) $$\int A t e^{-ct}\,dt = A\int \underbrace{t}_{u}\,\underbrace{e^{-ct}\,dt}_{dv} = A\Big[t\Big(\frac{e^{-ct}}{-c}\Big) - \underbrace{\int\Big(\frac{e^{-ct}}{-c}\Big)dt}_{+\frac{1}{c}\underbrace{\int e^{-ct}dt}_{\frac{e^{-ct}}{-c}}}\Big]$$

$u = t$, $\frac{du}{dt} = 1$, $du = dt$

$dv = e^{-ct}dt$, $v = \int e^{-ct}dt = \frac{e^{-ct}}{-c}$

$$= A\Big(-\frac{t}{c}e^{-ct} - \frac{1}{c^2}e^{-ct}\Big) + C \qquad = -Ae^{-ct}\Big(\frac{t}{c} + \frac{1}{c^2}\Big) + C$$

b) $$\int_0^\infty Ate^{-ct}dt = -Ae^{-ct}\Big(\frac{t}{c}+\frac{1}{c^2}\Big)\Big|_0^\infty = \lim_{a\to\infty} -Ae^{-ct}\Big(\frac{t}{c}+\frac{1}{c^2}\Big)\Big|_0^a$$

$$= \lim_{a\to\infty}\Big[-Ae^{-ca}\Big(\frac{a}{c}+\frac{1}{c^2}\Big) + Ae^{0}\Big(0+\frac{1}{c^2}\Big)\Big]$$

$$= -\frac{A}{c}\underbrace{\lim_{a\to\infty} \overset{\infty}{a}\,\overset{0}{e^{-ca}}}_{} - \frac{A}{c^2}\underbrace{\lim_{a\to\infty} e^{-ca}}_{0 \text{ since } c>0} + \frac{A}{c^2} = \frac{A}{c^2} = 1 \text{ if } A = c^2 \checkmark$$

$$\lim_{a\to\infty}\frac{a\to\infty}{e^{ca}\to\infty} = \lim_{a\to\infty}\frac{1}{ce^{ca}} = 0 \quad \text{(L'Hopital's rule)}$$

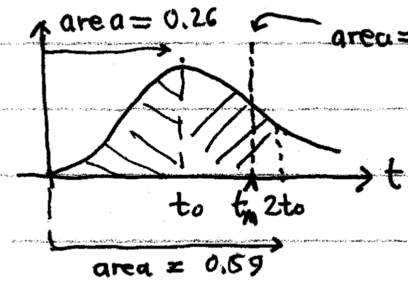
c) $$0 = \frac{d}{dt}(Ate^{-ct}) = A\big(1e^{-ct} + te^{-ct}(-c)\big) = A\underbrace{(1-ct)}e^{-ct}$$

$$\therefore\ 1-ct = 0 \to \boxed{t = \frac{1}{c}} = t_0$$

d) $$\int_0^{t_0} c^2te^{-ct}dt = -c^2e^{-ct}\Big(\frac{t}{c}+\frac{1}{c^2}\Big)\Big|_0^{1/c} = -c^2e^{-1}\Big(\frac{1}{c^2}+\frac{1}{c^2}\Big) + 1 = \boxed{1-2e^{-1} \approx 0.264}$$

e) $$\int_0^{2t_0} c^2te^{-ct}dt = -c^2e^{-ct}\Big(\frac{t}{c}+\frac{1}{c^2}\Big)\Big|_0^{2/c} = -c^2e^{-2}\Big(\frac{2}{c^2}+\frac{1}{c^2}\Big)+1 = \boxed{1-3e^{-2} \approx 0.594}$$

f)



area = 0.5 must be between them

so $\boxed{t_0 < t_M < 2t_0}$

or $\boxed{\frac{1}{c} < t_M < \frac{2}{c}}$

g) $$\frac{1}{2} = \int_0^{t_M} c^2\underbrace{t}_{\frac{u}{c}}\,e^{-\overbrace{ct}^{u}}\,\underbrace{dt}_{\frac{du}{c}} = \underbrace{\int_0^U ue^{-u}du}_{}$$

$u = ct$
$du = c\,dt$
$dt = du/c$
$t = u/c$

$t=0 \to u=0$
$t=t_M \to u = ct_M = U$

like setting $c=1$ and $t=u$.

U is called a "standard variable", having eliminated the parameter c used to set the scale for t.

set $c=1$:

h) $$\int_0^{t_M} te^{-t}dt = -e^{-t}(t+1)\Big|_0^{t_M}$$
$$= -e^{-t_M}(t_M+1)+1$$

Note $1 < t_M < 2$ from f) so

```
> plot([1/2, 1-exp(-t_M)*(t_M+1)],
       t_M = 0..2);
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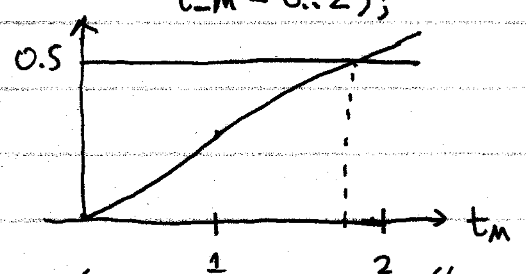


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> fsolve(1/2 = 1-exp(-t_M)*(t_M+1),
         t_M = 0..2);
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1.678346990

so $\boxed{t_M \approx 1.678}$