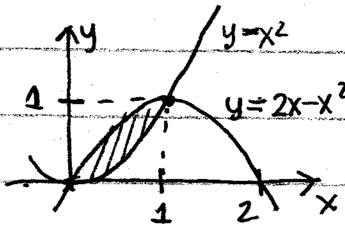
MAT1505-05/08 01S Final Exam Answers

① $y = \underbrace{2x-x^2}_{(2-x)x} = x^2 \to 2x^2+2x=0 \to x(x+1)=0$

$x=0$ or $x=-1$

$y=0 \quad y=1$.



$$A = \int_0^1 (2x-x^2)-x^2\,dx$$
$$= \int_0^1 2x-2x^2\,dx = 2\left(\frac{x^2}{2}\right)-2\left(\frac{x^3}{3}\right)\Big|_0^1$$
$$= x^2-\frac{2}{3}x^3\Big|_0^1 = 1-\frac{2}{3} = \boxed{\frac{1}{3}}$$

② $$f_{avg} = \frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2} x\cos 2x\,dx$$

$$\int \underbrace{x}_{u}\underbrace{\cos 2x\,dx}_{dv} = \underbrace{x}_{u}\underbrace{\left(\tfrac{1}{2}\sin 2x\right)}_{v} - \int \underbrace{\left(\tfrac{1}{2}\sin 2x\right)}_{v}\underbrace{dx}_{du}$$

$u=x \qquad dv = \cos 2x\,dx$

$\frac{du}{dx}=1 \qquad v=\int\cos 2x\,dx$

$du=dx \qquad = \frac{1}{2}\sin 2x$

$$= \tfrac{1}{2}x\sin 2x - \tfrac{1}{2}\left(-\tfrac{1}{2}\cos 2x\right)+C$$
$$= \tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x + C$$

$$f_{avg} = \frac{2}{\pi}\left(\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right)\Big|_0^{\pi/2}$$
$$= \frac{2}{\pi}\left[\frac{\pi}{4}\sin\pi + \frac{1}{4}\underbrace{\cos\pi}_{-1} - \left(0+\frac{1}{4}\underbrace{\cos 0}_{1}\right)\right] = \frac{2}{\pi}\left[-\frac{1}{4}-\frac{1}{4}\right] = \boxed{-\frac{1}{\pi}}$$

③ $$\int_{-1}^{0}\frac{x}{x-1}\,dx = \int_{x=-1}^{x=0}\frac{u+1}{u}\,du = \int_{x=-1}^{x=0}\left(1+\frac{1}{u}\right)du = u+\ln|u|\Big|_{x=-1}^{x=0}$$

$u = x-1 \to \cancel{x=u+1}$

$\frac{du}{dx}=1$

$du=dx$

$$= (x-1)+\ln|x-1|\Big|_{-1}^{0} = -1+\underbrace{\ln 1}_{0}-(-2+\ln 2)$$
$$= \boxed{1-\ln 2}$$

④ $$\int_1^2 \frac{\sqrt{1+x^2}}{x}\,dx = \int_2^5 \frac{z^{1/2}}{x}\left(\frac{dz}{2x}\right) = \boxed{\frac{1}{2}\int_2^5 \frac{z^{1/2}\,dz}{z-1}}$$

$z = 1+x^2 \quad x=1\to z=2$

$x=2\to z=5$

$\frac{dz}{dx} = 2x \qquad x^2 = z-1$

$\cancel{dz = 2x\,dx}$

$\frac{dz}{2x} = dx$

⑤ a) $$\frac{d}{dx}\left(-\frac{1}{2}e^{-x}(\cos x-\sin x)\right) = -\frac{1}{2}\frac{d}{dx}\left(e^{-x}(\cos x-\sin x)\right)$$
$$= -\frac{1}{2}\left(-e^{-x}(\cos x-\sin x)+e^{-x}(-\sin x-\cos x)\right)$$
$$= -\frac{1}{2}\left(-2e^{-x}\cos x\right) = e^{-x}\cos x.$$

b) $$\int_0^\infty e^{-x}\cos x\,dx = \lim_{a\to\infty}\int_0^a e^{-x}\cos x\,dx$$
$$= \lim_{a\to\infty} -\frac{1}{2}e^{-x}(\cos x-\sin x)\Big|_0^a = \lim_{a\to\infty} -\frac{1}{2}\underbrace{e^{-a}}_{\to 0}\underbrace{(\cos a-\sin a)}_{\text{each between } -1 \,\&\, 1} + \frac{1}{2}$$
$$= \boxed{\frac{1}{2}}$$

⑥ a) $s(t) = \int v(t)\,dt = \int t-\sin t\,dt = \boxed{\frac{t^2}{2}+\cos t+C}$

b) $0 = s(0) = 0+1+C \to C=-1$

$$\boxed{s(t) = \cos t+\frac{t^2}{2}-1}$$

⑦ a) $f(x) = (1+x)^{1/2} \qquad f(3) = 2$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f'(3) = \frac{1}{2}\left(\frac{1}{2}\right)$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2} \qquad f''(3) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2^3}\right)$

$f^{(3)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-5/2} \qquad f^{(3)}(3) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2^5}$

$f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1+x)^{-7/2} \qquad f^{(4)}(3) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{2^7}$

$$T_4(x) = \sum_{n=0}^{4}\frac{f^{(n)}(3)(x-3)^n}{n!}$$

$$T_4(x) = 2+\frac{1}{2^2}(x-3) - \frac{1}{2}\,\frac{1}{2^5}(x-3)^2 + \frac{1}{2\cdot 3}\,\frac{3}{2^8}(x-3)^3 - \frac{3\cdot 5\,(x-3)^4}{(2\cdot 3\cdot 4)\,2^{11}}$$

$\frac{1}{512} \qquad \frac{5}{16384}$

$$= 2+\frac{1}{4}(x-3)-\frac{1}{64}(x-3)^2+\frac{1}{2^9}(x-3)^3-\frac{5}{2^{14}}(x-3)^4$$

b) $$T_4(3.1) = 2+\frac{.1}{4}-\frac{.01}{64}+\frac{.001}{512}-\frac{5}{2^{14}}(.0001)$$

$= 2.000000$
$+ \;\; .025000$
$- \;\; .000156$
$+ \;\; .00000020 \leftarrow < \frac{1}{2}\times 10^{-5}$
$- \;\; .0000\,0003$

$\Big\}$ 2.0248 43750

c) If we dont think about roundoff error in the last digit, the first 3 terms should be enough for a result:

d) $\boxed{\sqrt{4.1} \approx 2.02484}$

(but if we add the next term it pushes to round up instead of round down leading to the correct result of 2.02485)

⑧ $a_n = (-1)^n\dfrac{x^n}{n(2n+1)4^{n+1}} \; (n>0), \quad a_0 = 0+C$

a) $S(0) = 0+C = C = 0 \to \boxed{C=0}$

b) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\dfrac{|x|^{n+1}}{(n+1)(2(n+1)+1)\,4^{n+1+1}}}{\dfrac{|x|^n}{n(2n+1)\,4^{n+1}}}$$

$$= \lim_{n\to\infty}\underbrace{\frac{|x|^{n+1}}{|x|^n}}_{|x|}\underbrace{\left(\frac{n}{n+1}\right)}_{\downarrow\, 1}\underbrace{\left(\frac{2n+1}{2n+3}\right)}_{\downarrow\, 1}\underbrace{\frac{4^{n+1}}{4^{n+2}}}_{\frac{1}{4}} = \frac{|x|}{4} < 1$$

$|x| < 4$

converges: $-4 < x < 4$

$x=4$: $a_n = \dfrac{(-1)^n 4^n}{n(2n+1)4^{n+1}} = \dfrac{(-1)^n}{4n(2n+1)}$

converges: alternating, abs value decreases to zero.

$x=-4$: $a_n = \dfrac{(-1)^n(-4)^n}{n(2n+1)4^{n+1}} = \dfrac{4^n}{n(2n+1)4^{n+1}} = \dfrac{1}{4n(2n+1)}$

large n $\to \dfrac{1}{8n^2}$ propto $p=2>1$ series, converges

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⑧ b) conclusion: $\boxed{\text{converges for } -4 \le x \le 4}$

⑨ a) $\dfrac{dy}{dt} = te^{y}$ separable

$$e^{-y}dy = t\,dt$$

$$\int e^{-y}dy = \int t\,dt$$

$$\frac{e^{-y}}{-1} = \frac{t^2}{2} + c_1$$

$$e^{-y} = -\frac{t^2}{2} - c_1$$

$$\ln(e^{-y}) = \ln\left(-\frac{t^2}{2} - c_1\right)$$

$$\overset{\shortparallel}{-y}$$

$$y = -\ln\left(-\frac{t^2}{2} - c_1\right)$$

$$0 = y(-1) = -\ln\left(-\frac{1}{2} - c_1\right)$$

$$1 = e^0 = e^{-\ln(-\frac{1}{2} - c_1)}$$

$$= \frac{1}{\left(-\frac{1}{2} - c_1\right)}$$

$$-\frac{1}{2} - c_1 = 1 \rightarrow c_1 = -\frac{1}{2} - 1 = -\frac{3}{2}$$

$$\boxed{y = -\ln\left(\frac{3}{2} - \frac{t^2}{2}\right)}$$

b) $$\frac{dy}{dt} = -\frac{1}{\left(\frac{3}{2} - \frac{t^2}{2}\right)}(0 - t) = \frac{t}{\frac{3}{2} - \frac{t^2}{2}}$$

$$te^{y} = te^{-\ln\left(\frac{3}{2} - \frac{t^2}{2}\right)} = \frac{t}{\frac{3}{2} - \frac{t^2}{2}}$$

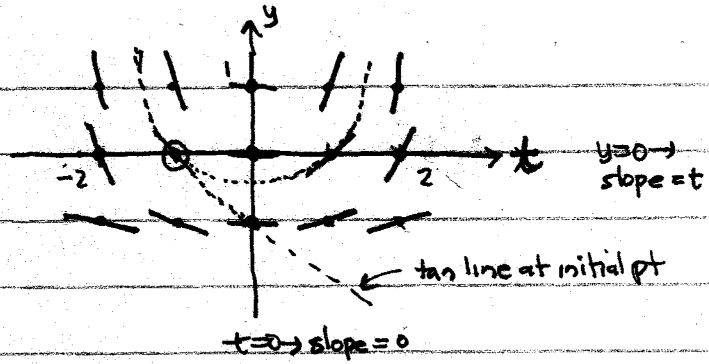
LHS = RHS ✓

$$y(-1) = -\ln\left(\frac{3}{2} - \frac{1}{2}\right) = -\ln 1 = 0 \ ✓$$

c) ln goes to $-\infty$ when input goes to 0^+:

$$\frac{3}{2} - \frac{t^2}{2} = 0 \rightarrow t^2 = 3 \rightarrow \boxed{t = \pm\sqrt{3}}$$

d)



e) $y(0) = -\ln\frac{3}{2} \approx \boxed{-.4055}$

f) $t_0 = -1 \qquad h = 1$

$y_0 = 0$

$t_1 = t_0 + h \qquad = -1 + 1 = 0$

$y_1 = y_0 + \underbrace{t_0 e^{y_0}}_{\text{slope}}\underbrace{h}_{\Delta t} \qquad = 0 + (-1)e^0 = -1$

$\boxed{y(0) \approx -1}$ for this approximation. (where tan line at initial pt crosses y-axis)