MATH1505-05/08 01 S TEST 2 Answers

① a) $f(x) = e^{-x}$ $\quad f(0) = 1$

$f'(x) = -e^{-x}$ $\quad f'(0) = -1$

$f''(x) = e^{-x}$ $\quad f''(0) = 1$

$\dots$

$f^{(n)}(x) = (-1)^n e^{-x}$ $\quad f^{(n)}(0) = (-1)^n$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = e^{-x}}$$

b) $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \dots$

$e^{-.1} = 1 - .1 + \frac{.1^2}{2} - \frac{.1^3}{6} + \frac{.1^4}{24} - \dots$

$$\left.\begin{aligned} = \ & 1.0000\ 00 \\ - \ & .1000\ 00 \\ + \ & .0050\ 00 \\ - \ & .0001\ 67 \end{aligned}\right\} .9048\ 375$$

d) $\longrightarrow \boxed{.9048}$

$+ .0000\ 04 < .5 \times 10^{-4}$ so 4

$- \dots$

c) terms should be enough

(next term in alternating series of decreasing positive terms estimates maximum error in absolute value)

e) See lower right.

② a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+1+3}} \cdot \frac{\sqrt{n+3}}{2^n |x-3|^n}$

$= \frac{2^{n+1}}{2^n} \frac{|x-3|^{n+1}}{|x-3|^n} \sqrt{\frac{n+3}{n+4}} = 2|x-3| \sqrt{\frac{1+3/n}{1+4/n}}$

$\xrightarrow[n\to\infty]{} 2|x-3| < 1$ for convergence (abs conv ratio test)

$|x-3| < \boxed{1/2 = R}$

$-1/2 < x-3 < 1/2$

$2.5 < x < 3.5$

at endpoints $x - 3 = \pm \frac{1}{2}$

$x-3 = \frac{1}{2}$: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}} \xrightarrow{\text{large } n} \sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$

$p = \frac{1}{2} < 1$ divergent p-series so original series diverges (limit comparison test).

$x-3 = -\frac{1}{2}$: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ converges by alt series test

since abs value terms decreasing to zero

or $\xrightarrow[n]{\text{large}} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{1/2}}$ alternating $p = 1/2$ series

again converges for same reason.

so interval of convergence: $\boxed{2.5 < x \le 3.5}$

③ $\displaystyle\int_1^{\infty} x e^{-x^2} dx$: $\displaystyle\int x e^{-x^2} dx = \int e^u \frac{du}{-2} = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2}$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2} du = x\,dx$

$\displaystyle\lim_{a\to\infty} -\frac{1}{2} e^{-x^2}\Big|_1^a = \lim_{a\to\infty} -\frac{1}{2} e^{-a^2} + \frac{1}{2} e^{-1} = \frac{1}{2} e^{-1}$

($e^{-a^2} \to 0$, decaying exponential)

integral converges so corresponding series converges.

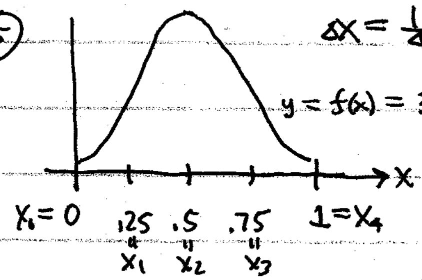
④ a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+n^2} \xrightarrow{\text{large } n} \sum_{n=1}^{\infty} \frac{1}{n^4}$ $\quad p = 4 > 1$ p-series so converges

so original converges (limit comparison test)

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{1+2n^2} \xrightarrow{\text{large } n} \sum_{n=1}^{\infty} \frac{1}{2}$ $\quad$ diverges since $a_n \not\to 0$

so original diverges (limit comparison test)

c) $\displaystyle\sum_{n=1}^{\infty} (\sqrt{2})^n$ is a geometric series with ratio $\sqrt{2} > 1$ so diverges.

⑤ 

$\Delta x = \frac{1}{4} = .25$

exact arithmetic approach

$$\int_0^1 f(x)\,dx \approx \frac{1}{3}\left(\frac{1}{4}\right)\left(f(0) + 4f\left(\tfrac{1}{4}\right) + 2f\left(\tfrac{1}{2}\right) + 4f\left(\tfrac{3}{4}\right) + f(1)\right]$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) 30 \left[\frac{0}{16}\left(\frac{1}{16}\right) + 4\left(\frac{1}{16}\right)\left(\frac{9}{16}\right) + 2\left(\frac{4}{16}\right)\left(\frac{4}{16}\right) + 4\left(\frac{9}{16}\right)\left(\frac{1}{16}\right) + \left(\frac{16}{16}\right)\left(\frac{0}{16}\right)\right]$$

$$= \frac{5}{2}\left[\frac{1}{16^2}\right]\left[4\cdot 9 + 4\cdot 8 + 4\cdot 9\right] = \frac{5}{2}\left(\frac{1}{8^2}\right)(9+8+9)$$

$$= \frac{5\cdot 13}{8^2} = \frac{65}{64} = \boxed{1.015625 = S_4}$$

OR decimal approach:

$$= \frac{.25}{3}\left[f(0) + 4f(.25) + 2f(.5) + 4f(.75) + f(1)\right]$$

$$= \frac{.25}{3}(30)\left[0^2(1^2) + 4(.25)^2(.75)^2 + 2(.5)^2(.5)^2 + 4(.75)^2(.25)^2 + 1^2(0^2)\right]$$

$= (\text{technology}) = \boxed{1.015625 = S_4}$

The percentage error compared to the exact value of 1 is clearly about $\boxed{1.56\%}$.

(①e) $\displaystyle x e^{-x} = x \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n!}}$

$\left.\begin{aligned} m &= n+1 \\ m-1 &= n \\ n=0 &\to m=1 \end{aligned}\right\} \to \boxed{= \displaystyle\sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{(m-1)!}}$