MATH 2705 005 TEST 2 ANSWERS

① a) $B^TB = \begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\begin{bmatrix}1&3\\2&2\\3&1\end{bmatrix} = \begin{bmatrix}1+4+9 & 3+4+3\\3+4+3 & 9+4+1\end{bmatrix} = \begin{bmatrix}14&10\\10&14\end{bmatrix}$.

$2\times3 \quad 3\times2 \to 2\times2$

b) $AA^{-1} = \begin{bmatrix}1&2&3\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&-2&5\\0&1&-4\\0&0&1\end{bmatrix} = \begin{bmatrix}1+0 & -2+2+0 & 5-8+3\\0 & 0+1+0 & 0-4+4\\0 & 0 & 0+1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ ✓

c) $AX=B$: $\begin{bmatrix}1&2&3\\0&1&4\\0&0&1\end{bmatrix}X = \begin{bmatrix}1&3\\2&2\\3&1\end{bmatrix}$

$3\times3 \quad 3\times? \quad 3\times2$

$\to 3\times?$

X must be a 3x2 matrix, 3 rows to be able to multiply A on the right and 2 columns to produce a product with 2 columns.

d) $A^{-1}(AX=B) \to IX = A^{-1}B \to X = A^{-1}B = \begin{bmatrix}1&-2&5\\0&1&-4\\0&0&1\end{bmatrix}\begin{bmatrix}1&3\\2&2\\3&1\end{bmatrix} = \begin{bmatrix}1-4+15 & 3-4+5\\0+2-12 & 0+2-4\\0+3 & 0+1\end{bmatrix}$

$= \begin{bmatrix}12&4\\-10&-2\\3&1\end{bmatrix}$.

② a) $\begin{bmatrix}2&1&0&-1\\1&0&2&1\\2&0&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}2\\3\\6\end{bmatrix}$ (A, $\vec{x}$, $\vec{b}$)

$C = \begin{bmatrix}2&1&0&-1&2\\1&0&2&1&3\\2&0&1&5&6\end{bmatrix}$ (b) $\to \begin{bmatrix}1&0&2&1&3\\2&1&0&-1&2\\2&0&1&5&6\end{bmatrix} \to \begin{bmatrix}1&0&2&1&3\\0&1&-4&-3&-4\\0&0&-3&3&-6\end{bmatrix}$

$\to \begin{bmatrix}1&0&2&1&3\\0&1&-4&-3&-4\\0&0&1&-1&2\end{bmatrix} \to \begin{bmatrix}1&0&0&3&-1\\0&1&0&-7&4\\0&0&1&-1&2\end{bmatrix}$ (B B B F)

$x_1 + 3x_4 = -1$, $x_2 - 7x_4 = 4$, $x_3 - x_4 = 2$, $x_4 = t_1$

$x_1 = -1-3t_1$, $x_2 = 4+7t_1$, $x_3 = 2+t_1$, $x_4 = t_1$

③ a) $\begin{vmatrix}1&1&1&1\\-1&1&-1&1\\1&1&-1&-1\\-1&1&1&-1\end{vmatrix} = \begin{vmatrix}1&1&1&1\\0&2&0&2\\0&0&-2&-2\\0&2&2&0\end{vmatrix}$ (3 addrowops = pivot)

$= 2(-2)(2)\begin{vmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\\0&1&1&0\end{vmatrix}$ (factor out 3 leading entries)

$= -8\begin{vmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\\0&0&1&-1\end{vmatrix}$ (1 addrowop)

$= -8\begin{vmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\\0&0&0&-2\end{vmatrix}$ (1 addrowop)

$= (-8)(1)(1)(1)(-2) = 16$ (det triangular matrix = prod diag values.)

b) since $\det(A) \neq 0$ it has an inverse A^{-1}, so

$A\vec{x} = \vec{b}$ has the unique solution $A^{-1}(A\vec{x}) = A^{-1}\vec{b} \to \vec{x} = A^{-1}\vec{b}$.

④ a) $x_1 + 2x_2 + 3x_4 = 0$

$x_3 - 4x_4 = 0$

$0 = 1$ ← inconsistent system, no solutions.

$0 = 0$

b) $x_1 + 2x_2 + 3x_4 = 0 \to x_1 = -2x_2 - 3x_4 = -2t_1 - 3t_2$

$x_3 - 4x_4 = 0 \to x_3 = 4x_4 = 4t_2$

$x_6 = 0 \to x_6 = 0 = 0$

B vars → F vars: $x_2 = t_1$, $x_4 = t_2$, $x_5 = t_3$

$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix} = \begin{bmatrix}-2t_1-3t_2\\t_1\\4t_2\\t_2\\t_3\\0\end{bmatrix}$

c) If $B = \text{rref}(A)$, then B has 3 nonzero rows so $\boxed{\text{rank}(A) = 3}$.